Math for the Physical Sciences

by

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Notice

This is a "lecture note" style review guide, originally designed to support my personal teaching activities at Duke University. It is freely available in its entirety to all the students of the world in a downloadable PDF form, or it can be be read online at:

http://www.phy.duke.edu/~rgb/Class/intro_math_review.php

and will be made available in an inexpensive print version via Lulu press as soon as it is in a sufficiently polished and complete state.

I make this available for free *for personal use only* so that the text can be used by students all over the world regardless of their means or ability to pay. Nevertheless, I am hoping that students who truly find this work useful will purchase an (inexpensive) copy of this text, if only to help subsidize me while I continue to write more inexpensive textbooks.

Be warned: As a "living" document that I actually use to teach, these notes may have errors of omission or commission. Expect them to change without warning as I add content or correct errors. Purchasers of any paper version should be aware of its probable imperfection and be prepared to either live with it or mark up their *own* copies with corrections or additions as need be (in the lecture note spirit) as I do mine. The text has generous margins, is widely spaced, and contains a number of scattered blank pages for students' or instructors' own use to facilitate this.

Note well that this is an "odd" book in that it isn't intended to be used as a textbook *ever* for *any* course even though it may well prove to be better than any real textbook for learning or relearning the material it covers quickly. This is in part because this book has *no homework* problems in it. There are no exercises. There is no possibility of a student being given "the assignment on page 23" to complete by Monday.

Yet to learn something it is essential to *do* something and not just read a book or listen to a lecture. Tough. Maybe one day I'll write up an associated book of problems. Or (if you're using it to teach or learn math anyway) you can always make up problems of your own. Or best of all, it can be used for its intended purposes – to be the book in your left hand while your right is working out homework problems in *something else* – physics, economics, chemistry, or even algebra, trigonometrics, calculus. Math as mindless manipulation of empty symbols doesn't appeal that much even to most mathematicians. Math as a process of *reasoning* about *problems* with *meaning* on the other hand, can be a real pleasure!

I cherish good-hearted communication from students or other instructors pointing out errors or suggesting new content (and have in the past done my best to implement many such corrections or suggestions).

Books by Robert G. Brown

Physics Textbooks

- *Equations du Jour I & II*

  A lecture note style textbook intended to support the teaching of introductory physics.

- *Mathematics for the Physical Sciences*

  A review of all the essential mathematics needed by students taking introductory physics or other physical science courses at the college level. However, it is written to be a gangbusters review for the SATs, for students struggling in a calculus or precalculus class, or even for smart students in middle school who want to "instantly" learn the essential results from all of high school math without all the pain and suffering.

- *Classical Electrodynamics II*

  A set of lecture notes intended to support the second semester of a two semester course based on J. D. Jackson's book of the same name, although it can also stand alone as a textbook.

Computing Books

- *How to Engineer a Beowulf Cluster*

  An online classic for years, this is the print version of the famous free online book on cluster engineering. It too is being actively rewritten and developed, no guarantees, but it is probably still useful in its current incarnation.

Fiction

- *The Book of Lilith*

  ISBN: 978-1-4303-2245-0
  Web: http://www.phy.duke.edu/~rgb/Lilith/Lilith.php

  Lilith is the *first* person to be given a soul by God, and is given the job of giving all the things in the world souls by loving them, beginning with Adam. Adam is given the job of making up rules and the definitions of sin so that humans may one day live in an ethical society. Unfortunately Adam is weak, jealous, and greedy, and insists on being on *top* during sex to "be closer to God".

Lilith, however, refuses to be second to Adam or anyone else. *The Book of Lilith* is a funny, sad, satirical, uplifting tale of her spiritual journey through the ancient world soulgiving and judging to find at the end of that journey – herself.

- *The Fall of the Dark Brotherhood*

  ISBN: 978-1-4303-2732-5
  Web: http://www.phy.duke.edu/~rgb/Gods/Gods.php

  A straight-up science fiction novel about an adventurer, Sam Foster, who is forced to flee from a murder he did not commit across the multiverse. He finds himself on a primitive planet and gradually becomes embroiled in a parallel struggle against the world's pervasive slave culture and the cowled, inhuman agents of an immortal of the multiverse that support it. Captured by the resurrected clone of its wickedest agent and horribly mutilated, only a pair of legendary swords and his native wit and character stand between Sam, his beautiful, mysterious partner and a bloody death!

Poetry

- *Who Shall Sing, When Man is Gone*

  Original poetry, including the epic-length poem about an imagined end of the world brought about by a nuclear war that gives the collection its name. Includes many long and short works on love and life, pain and death.

  > Ocean roaring, whipped by storm
  > in damned defiance, hating hell
  > with every wave and every swell,
  > every shark and every shell
  > and shoreline.

- *Hot Tea!*

More original poetry with a distinctly Zen cast to it. Works range from funny and satirical to inspiring and uplifting, with a few erotic poems thrown in.

> Chop water, carry
> wood. Ice all around,
> fire is dying. Winter Zen?

All of these books can be found on the online Lulu store here:

http://stores.lulu.com/store.php?fAcctID=877977

Both *The Book of Lilith* and *The Fall of the Dark Brotherhood* are also available on Amazon, Barnes and Noble and other online booksellers, and one day from a bookstore near you!

Contents

Preface

If you're reading this, I'm going to presume that you are a student in high school or college preparing to take calculus, or physics, or perhaps chemistry or engineering. You've had lots of math – algebra, geometry, trigonometry, perhaps a few semesters of calculus.

However, you've never really had to *use* all this math to reason with. When presented with a page of integrals you can (possibly with a few minutes to review) manage to work through most of them correctly, but when presented with a problem in words and pictures, especially one that depends on relations you're struggling to master (like the laws of physics) you have no idea how to proceed to *reduce* the problem to just doing some particular integral.

There are also a bunch of things that you just plain don't know. Maybe they were covered one time in one course that you've taken. Maybe *your* book, or *your* class, or *your* teacher just failed to cover them. Maybe you were absent that day. Who knows?

However it came about, you find yourself in a course where you *really need* to know those things, to have them at the tip of your fingers, ready to pour out intelligently onto paper as you solve problems that are difficult enough if you *do* know well all of the things required to solve them.

This short book is for you. It is written in a lecture note, summary/review style that should make it fairly easy to find things that you need to know quickly, and filled with the actual derivations and short explanations that support the material so that you can actually understand it and not just memorize it. If you keep it handy – on your laptop, in your backpack – it can be a quick and easy way to look things up or review them while doing your homework in various science courses or studying for exams.

Chapter 1

Introduction

Here are some of the mathematical topics that are covered in this book:

- **Numbers**

  Integers, real numbers, complex numbers, prime numbers, important numbers, the algebraic representation of numbers. The study of science is all built upon numbers, and physics was *co-developed* with much of modern mathematics with numbers themselves as its ultimate foundation.

  Interestingly, few students are taught much about numbers per se before they reach college, and yet they play a key role in nearly all science and engineering.

- **Algebra**

  Algebra is the *symbolic manipulation* of numbers according to certain rules. Algebra plays a crucial role in nearly all modern science, as it is the *language of reason* used to describe *general relationships* instead of specific instances. Through high school a student can often get by just doing simple arithmetic to (for example) compute an average speed going from point A to point B in time t. In college one stops caring so much about the *particular* distance between these two points and the *particular* time of the journey and learns to reason with symbols that embrace all possible beginning and end point and all possible time intervals.

Practically speaking, algebra is nearly always used to (for example) solve for a particular desired symbolic quantity (or quantities) that are "unknown" terms of others that are "known". However, it is also used in the process of learning itself to derive *new relationships* hidden in things we already know and understand, and hence extend our understanding without needing to rely on memorization!

- **Coordinate Systems and Vectors**

  The difference between (say) pure mathematics and physics or chemistry is that in the sciences quantities are nearly always associated with a *system of units*, a *coordinate system* where a given algebraic parameter for mass or position or time can take on many possible values that carry with them a characteristic scale and relationship to other quantities.

  Perhaps one of the most fundamental of these relationships is that of location. As we struggle to create a *mental conceptual model* of the external Universe that we perceive through our senses we will rely heavily on the ability to create a *map* between that outside world and our internal conceptual model. The coordinates of space and time (as well things like mass, charge, spin and more that describe objects *in* space-time) *are* that internal mental symbolic model.

  To support our learning about the Universe, we therefore need to be familiar with cartesian coordinates in 1 dimension, cartesian and plane polar coordinates in 2 dimensions, cartesian, cylindrical and spherical coordinate systems in 3 dimensions. We need to be able to at least *think* about time as a dimension – either an "independent" one-dimensional object that serves as the fundamental parameter in a dynamical description of nature or a literal fourth dimension when one starts to study relativity theory.

  In association with these coordinates, we'll need to understand things like scalars and vectors (including vector addition, subtraction, inner (dot) product of vectors, outer (cross) product of vectors, length of a vector).

  Scalars and vectors are the first two ranks of a systematic extension of coordinates and numbers that also includes matrices and *tensors* of higher rank. You probably won't need to know much about tensors in introductory courses, but we'll include a short discussion of them for more advanced students of physics – enough to at least get you started.

- **Useful Functions**

  There are an infinite number of functional relationships between one or more coordinates (as a domain) and one or more other coordinates (as a range). Wow! That's a lot!

  Fortunately, only a very finite *handful* of functions from that infinity turn out to be really important to the understanding of introductory science. Fortunately, most of those functions are pretty simple (and simply pretty!). We will review the ones you absolutely need to know to make sense of the world through the eyes of science.

  These include polynomials (of course), where for a variety of reasons we'll be most interested in the first three polynomial functions – constants, straight lines, and quadratics – more than we will be interested in (say) polynomials with powers of 3 or more present.

  Trigonometric functions, exponential functions, and logarithms are perhaps even more important, and are not actually distinct objects. There is a beautiful relationship between *complex numbers* and trig functions such as sine, cosine and tangent. The *exponential of a complex number* turns out to be a very important idea as well. We review this in a way that hopefully will make working with these functions quite easy.

- **Differentiation**

  Physics is all about related rates, and calculus was *invented* (by Newton and Leibnitz) to do physics. Chemistry involves many rate equations. Biology, geology, even social sciences like economics all involves calculus and related rate equations, or *differential relations*. We quickly review what differentiation *is*, and then present, sometimes with a quick proof, a table of derivatives of functions that you should know to make learning physics or other sciences at this level straightforward. Basically, this includes all the functions in the previous section, plus some methodology.

- **Integration**

Integration is basically antidifferentiation or summation. Since many physical relations involve summing, or integrating, over extended distributions of mass, of charge, of current, of fields, we present a table of integrals (some of them worked out for you in detail so you can see how it goes). Again, most of what you need to know involves the handful of important functions plus methodology that lets you do "interesting" variations thereof.

- **Miscellaneous Stuff**

  This isn't everything, of course. There are things you just need to know that don't fit neatly into the categories above. We'll try to collect and summarize some of this in a catchall chapter to fill in the cracks.

Chapter 2

Numbers

2.1 Integers

This is the set of numbers[1] :

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

that is pretty much the first piece of mathematics any student learns. The part of the set beginning with 1,2,3,... (called the *natural numbers* are used to *count*, initially to count objects such as pennies or marbles.

Integers are typically defined along with a set of operations known as *arithmetic*[2] . The well-known operations of arithmetic are addition, subtraction, multiplication, and division. Subtraction can always be viewed as the addition of a positive and negative number; it is not really a separate operation. One rapidly sees that the set of integers is not *closed* with respect to them. The sum, difference, or product of two integers is an integer, but an integer divided by an integer is not necessarily an integer.

Natural numbers greater than 1 in general can be factored into a representation in *prime numbers*. For example $45 = 2^0 3^2 5^1 7^0 ...$ or $56 = 2^3 3^0 5^0 7^1 11^0 ....$ Prime numbers are important in *number theory* (the study of numbers in mathematics) but are not so important in the sciences.

Integers can in general also be factored into primes, with two small mod-

[1] Wikipedia: http://www.wikipedia.org/wiki/number.
[2] Wikipedia: http://www.wikipedia.org/wiki/arithmetic.

ifications. First, negative integers will always carry a factor of -1 times the prime factorization of its absolute value. Second, 0 times anything is 0, so it (and the number ± 1) are generally excluded from this factorization process.

Integer arithmetic has a number of lovely and useful algebraic properties – associativity, commutivity, distributivity – but we will defer putting them down until we have defined the *real* numbers below, since all integers are real numbers and "inherit" these properties from the reals.

Similarly the *base* that is nearly always used for integers and rational numbers is base 10, a representation of integers as a sum of powers of ten: $3012 = 3 \times 10^3 + 0 \times 10^2 + 1 \times 10^1 + 2 \times 10^0$.

The use of bases other than 10 aren't terribly common in science, but base 2 (binary) is very important in computer science. In base two a number is similarly expressed in powers of two: $21 = 16 + 4 + 1 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 10101$.

2.2 Rational Numbers

If one takes two integers a and b and divides a by b to form $\frac{a}{b}$, the result will often *not* be an integer. For example, 1/2 is not an integer, nor is $1/3, 1/4, 1/5...$, nor $2/3, 4/7, 129/37$ and so on, although it may be; $4/2 = 2$ *is* an integer, for example. These numbers are all the *ratios* of two integers and are hence called *rational numbers*[3] .

The rational numbers *do* close under division, with the single exception of division by zero, which is undefined. They can therefore be *factored* and form a *division algebra*.

Rational numbers in physics are generally expressed in base 10, which simply extends the representation for integers above with negative powers of ten: $3012.694 = 3\times10^3+0\times10^2+1\times10^1+2\times10^0+6\times10^{-1}+9\times10^{-2}+4\times10^{-3}$.

Rational numbers expressed in a base have an interesting property. Dividing one out produces a finite number of non-repeating digits, followed by a finite sequence of digits that repeats cyclically forever, for example $1/6 = 0.1666...$. Note that finite precision decimal numbers are precisely

[3]Wikipedia: http://www.wikipedia.org/wiki/rational number.

those that are terminated with an infinite string of 0 digits – this is the special class of numbers over which we do much of our arithmetic especially on digital computers.

If all rational numbers have digit strings that eventually cyclically repeat, what about all numbers whose digit strings do *not* cyclically repeat? These numbers are *not* rational, and cannot be expressed as the ratio of two finite integers, since the integers would require an infinite number of digits and hence would be infinite themselves.

2.3 Irrational Numbers

An irrational number[4] is one that *cannot be written* as a ratio of two integers e.g. a/b. Most proofs that any given number is irrational involve assuming that it can be so written and showing that this leads to a contradiction.

It is quite easy to show that e.g. $\sqrt{2}$ is irrational using this method, although there are other numbers of "interest" in physics and science where the property of irrationality is not so easy to prove. Two irrational numbers that are of great importance in physics are $e = 2.718281828...$ and $\pi = 3.141592654...$.

Whenever we *compute* a number answer we *must* use rational numbers to do it, most generally a finite-precision decimal representation. For example, 3.14159 may *look* like π, an irrational number, but it is really $\frac{314159}{100000}$, a rational number that *approximates* π to six significant figures.

For that reason we will often carry important irrationals along with us in computations as *symbols* and only evaluate them numerically at the end. This often yields more satisfactory answers. Consider $\sqrt{2} * \sqrt{2} = 1.414 * 1.414 = 1.999396$ in a decimal representation. This answer is not *exactly* 2.

Also, we work quite often with functions that yield a rational number when an irrational number is used as an argument, e.g. $\cos(\pi) = -1$. If we did finite-precision arithmetic we might get instead $\cos(3.14) = -0.999999$ which is not *exactly* -1.

These computational errors aren't usually terribly important for small

[4]Wikipedia: http://www.wikipedia.org/wiki/irrational number.

one or two step calculations, but in a long-running computer program they can easily add up! However, unless we can recognize a rational answer in an expression that contains irrational numbers, we will more or less have to work with decimal (rational) approximations and do our best to control these "round-off" errors. Fortunately, we can make the difference between an irrational number and a rational approximation to it as small as we like by just adding more digits to the latter.

There are lots of nifty truths regarding irrational and irrational numbers. For example, in between any two rational numbers lie an *infinite* number of *irrational* numbers. This is a "bigger infinity"[5] than *just* the countably infinite number of integers or rational numbers, which actually has some important consequences in physics. Mostly, however, we will be pretty happy working with a truncated rational decimal representation with a finite number of "significant digits".

2.4 Real Numbers

The union of the irrational and rational numbers forms the *real number line*[6] . The properties of irrationals, rationals, and integers are primarily inherited from the fact that they are all particular kinds of *real numbers*. The set of real numbers is closed under the arithmetical operations of addition/subtraction (again, really the same thing with subtraction just being the addition of negative numbers), multiplication and division (where one must exclude division by zero as undefined).

Later we will discuss the *algebra* of operations based on real numbers, but here we can list some of the *arithmetical properties* of the reals. For addition:

$$a + (b + c) = (a + b) + c \quad \text{associativity} \tag{2.1}$$

$$a + b = b + a \quad \text{commutativity} \tag{2.2}$$

$$a + 0 = a \quad \text{identity} \tag{2.3}$$

$$a + (-a) = 0 \quad \text{inverse} \tag{2.4}$$

[5]Wikipedia: http://www.wikipedia.org/wiki/infinity.
[6]Wikipedia: http://www.wikipedia.org/wiki/real line.

Similarly, for multiplication:

$$
\begin{aligned}
a * (b * c) &= (a * b) * c \quad \text{associativity} && (2.5)\\
a * b &= b * a \quad \text{commutativity} && (2.6)\\
a * 1 &= a \quad \text{identity} && (2.7)\\
a * \frac{1}{a} &= a * a^{-1} = 1 \quad \text{inverse} && (2.8)
\end{aligned}
$$

Note that division can be defined as multiplication by the inverse of a number, just as subtraction is the addition of the additive inverse of a number.

For combinations of multiplication and addition, we can add:

$$a * (b + c) = a * b + a * c \quad \text{distributivity} \tag{2.9}$$

2.5 Complex Numbers

We now have a bit of a bootstrapping problem. Complex numbers arise out of the reals when we consider the *exponentiation* process, which we won't really talk about until later, in the chapter on Algebra. If you have no idea what $b = a^n$ or $a = b^{1/n}$ mean and aren't at all familiar with rules such as $c = a^n b^n = (ab)^n$ and $c = a^n a^m = a^{(n+m)}$ then you should probably pause here and take a quick trip over to where this is covered and come back. We'll wait for you.

Fine. If you are now reading on, I will *assume*, dear reader, that you are now passingly familiar with the concept of the square root $a^{1/2} = \sqrt{a}$. Consider the square root operation applied to the real numbers. The square root operation (and many other non-integer-power operations) *do not close* under the reals. Since any number (positive, negative, or zero) squared is a *non-negative* number, we cannot find the square root of e.g. -1 among the reals.

We then have a choice – we can say fine, we will simply restrict the square root operation so that its domain is non-negative and call attempts to take the square root of a negative number *undefined* – as you probably did in your *introductory* algebra class back in high school – or we can be a bit more sophisticated and *imagine* a number (that clearly isn't "real") whose square is negative.

Since algebra is going to lead us to expressions containing square roots of arguments that can be negative *quite a lot* in science, it turns out to be useful to do the latter. Indeed, we will find it more than just "useful" – it will be *essential* later on down the road, even in introductory physics or math classes to understand this natural extension of the real numbers.

You are probably familiar with the naive definition of the *unit imaginary number*[7] as the square root of -1:

$$i = +\sqrt{-1} \tag{2.10}$$

This definition is common but slightly unfortunate. If we adopt it, we have to be careful *using* this definition in algebra (using the rules for exponentiation developed below) or we will end up proving any of the many variants of the following:

$$-1 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{-1 \cdot -1} = \sqrt{1} = 1 \tag{2.11}$$

Oops.

A *better* definition for i that it is just the algebraic number such that:

$$i^2 = -1 \tag{2.12}$$

and to leave the square root bit out. Thus we have the following cycle:

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= (i^2)i = -1 \cdot i = -i \\ i^4 &= (i^2)(i^2) = -1 \cdot -1 = 1 \\ i^5 &= (i^4)i = i \\ &\ldots \end{aligned} \tag{2.13}$$

where we can use these rules to do the following sort of simplification:

$$+\sqrt{-\pi b} = +\sqrt{i^2 \pi b} = +i\sqrt{\pi b} \tag{2.14}$$

where we never actually write $i = \sqrt{-1}$ except in the sense of $i = \sqrt{-1} = \sqrt{i * i}$.

[7] Wikipedia: http://www.wikipedia.org/wiki/imaginary unit.

This sounds really complicated and awful but isn't. It is actually really easy to avoid the "bad step" by simply never working *backwards* from $i \rightarrow \sqrt{-1}$ in algebra. One can always pull an i *out* from under a radical (and change the sign of its contents) but it is very, very dangerous to put an i *back in* under a radical as a change in sign.

With this definition, we can define an arbitrary complex number z as:

$$z = x + iy = z_r + iz_i \tag{2.15}$$

where x and y (or z_r, the "real part of z" and z_i, the "imaginary part of z") are both real numbers. We will work out the details of algebra for complex numbers later in this review. Here we only wish to note that they satisfy *exactly the same basic properties* with respect to arithmetic operations as the reals.

To add two complex numbers $a = a_r + ia_i$ and $b = b_r + ib_i$, we use the rule:

$$a + b = (a_r + b_r) + i(a_i + b_i) \tag{2.16}$$

That is, we *add the real and imaginary parts separately!* To multiply them we just use the usual product rule for sums:

$$\begin{aligned} c &= ab \\ &= (a_r + ia_i) * (b_r + ib_i) \\ &= a_r b_r + ia_i b_r + ia_r b_i + (i * i)a_i b_i \\ &= (a_r b_r - a_i b_i) + i(a_i b_r + a_r b_i) \\ &= c_r + ic_i \end{aligned} \tag{2.17}$$

where $c_r = a_r b_r - a_i b_i$ and $c_i = a_i b_r + a_r b_i$.

We define the *complex conjugate* of a complex number $z = x + iy$ as $z^* = \bar{z} = x - iy$ (both notations are used, so I indicate them both). The point of defining the complex conjugate of a complex number is that:

$$|z|^2 = zz* = z * z = x^2 + y^2 \tag{2.18}$$

is a *real* number that is called the *amplitude* or *magnitude* of a complex number – its "size". It is a measure of the *distance from the complex origin* of the complex number. We'll examine the geometry of this below when we examine the algebraic properties of complex numbers in more detail.

For addition, if a, b, and c are arbitrary *complex* numbers:

$$a + (b + c) = (a + b) + c \quad \text{associativity} \tag{2.19}$$
$$a + b = b + a \quad \text{commutativity} \tag{2.20}$$
$$a + 0 = a \quad \text{identity} \tag{2.21}$$
$$a + (-a) = 0 \quad \text{inverse} \tag{2.22}$$

where e.g. $a = a_r + ia_i$ implies $-a = -a_r - ia_i$.

Similarly, for multiplication:

$$a * (b * c) = (a * b) * c \quad \text{associativity} \tag{2.23}$$
$$a * b = b * a \quad \text{commutativity} \tag{2.24}$$
$$a * 1 = a \quad \text{identity} \tag{2.25}$$
$$a * \frac{1}{a} = a * a^{-1} = 1 \quad \text{inverse} \tag{2.26}$$

just as before.

The only tricky part of division is that one has to perform a little ritual to put the inverse of a complex number back into the form of a complex number. We'll go over this when we do algebra, but it doesn't hurt to put it here as well. It is:

$$\begin{aligned} a = \frac{1}{z} & \\ &= \frac{1}{x+iy} \\ &= \frac{1}{x+iy}\frac{x-iy}{x-iy} \\ &= \frac{x-iy}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \end{aligned} \tag{2.27}$$

or $a_r = \frac{x}{x^2+y^2}$, $a_i = -\frac{y}{x^2+y^2}$. This clever trick (multiplying by z^*/z^*) makes the denominator *real* so that the expression can be simplified into real and imaginary parts cleanly.

For combinations of multiplication and addition, we obviously still have

$$a * (b + c) = a * b + a * c \quad \text{distributivity} \tag{2.28}$$

as well.

The reals *inherit* their properties (as you can see) from the complex numbers. Later we'll see just how powerful complex numbers can be in helping to solve problems.

At this point you should begin to have the feeling that this process of generating supersets of the number system(s) we already have figured out up to any given point will never end. You would be right. You are seeing the first steps of a *hierarchy* of number systems and algebras. Complex numbers are all one usually needs in *introductory* courses in physics or engineering (so you can relax, we're going to stop here) but further extensions (e.g. quaternions[8] or generalized *geometric algebras*[9] the extensions are actually very useful in *advanced* physics.

This concludes our discussion of *numbers* per se. When introducing and discussing numbers, we found it very convenient to let various *symbols* stand for "example" numbers of each different type. In English we might say: "We can always add two real numbers in either order and get the same result" but that's a whole lot of writing. It was much more compact to say: $a+b=b+a$ where a and b can stand for *any* real number. We can then extend this same relation to complex numbers by saying that a and b can be complex (or quaternionic, or whatever).

This is an example of the power of *algebra*. Algebra is basically the process of letting symbols stand for numbers, and then deducing and defining rules and processes that let us take the simple definitions of numbers of different kinds above and work all sorts of symbolic magic on them. There is something sublime, almost ethereal about algebra. It is at once *a system of reason* – a way of connecting things we already "know" to new things that we discover by means of its rules – and *the natural language of science*.

Science, after all, isn't satisfied with knowledge expressed in English as "If you let go of a penny, it will fall down". It wants an algebraic *Law of Universal Gravitation* that simultaneously explains (after a fashion) *why* the penny falls down while defining what "down" is, while providing you with relations that permit you to compute quite exactly what the behavior of the penny is as it falls. One is qualitative, and doesn't tell you what the

[8]Wikipedia: http://www.wikipedia.org/wiki/quaternion.
[9]Wikipedia: http://www.wikipedia.org/wiki/geometric algebra.

penny would do on the Moon, where down may well be up relative to the Earth, or how fast it falls, or why it falls. The other (if accepted as a good approximation to truth) is general and quantitative – it works for pennies, nickels, dimes, and rocks.

Let us therefore take a look at algebra and review everything about it that you are likely to need to know.

Chapter 3

Algebra

Algebra[1] is a *reasoning process* that is one of the fundamental cornerstones of physical science and engineering. As far as we are concerned, it consists of two things:

- Representing *numbers* (quantities) of one sort or another with *symbols*.

  In the sciences we don't only use symbolic representation for "unknown" parameters – we will often use algebraic symbols for numbers we know or for parameters whose value is actually given in problems. In fact, with only a relatively few exceptions, we will *prefer* to use symbols as much as we can to permit our algebraic manipulations to eliminate as much eventual *arithmetic* (computation involving actual numbers) as possible. We hate doing arithmetic, right?

- Performing a sequence of *algebraic transformations* of a set of equations or inequalities to convert it from one form to another (desired) form.

  The permitted transformations are generally based on the set of arithmetic operations defined over the field(s) of the number types being manipulated that we talked about above and extended by means of suitable definitions that help us "compress" our representation a little bit.

That's it. It is really – simple – simple enough to be the *best* way of reasoning about nearly anything. Even English is, in some sense, a system of algebraic

[1] Wikipedia: http://www.wikipedia.org/wiki/algebra.

reasoning, with symbols standing for things. We're just going to streamline the process and make it more consistent than English usually is.

There are two ways students are usually taught algebra. One is by being given all sorts of problems of the form:

1. Given $3t^2 + 32t + 20 = 0$, find the two roots t_1 and t_2.
2. Given a straight line $y(x)$ that passes through the points $(x, y) = (2, 0)$ and $(x, y) = (5, 6)$, find a and b in the equation $y(x) = ax + b$.
3. If $A = 9$, what is $V = A^{3/2}$?

Of course, students aren't given only three such problems; they're given three hundred! And each problem is usually given only in groups of other problems of the same kind, to drill the student in some aspect of algebra. This is bo-ring and appears to be pointless to many young students. Even if they dutifully do all of the exercises, they quickly forget.

The second is by being given problems in words, like:

1. At time $t = 0$ a truck is observed 20 meters down the road from a stoplight. A policeman clocks it with radar travelling at 32 m/sec and accelerating at a rate of 6 m/sec^2. Assuming that its acceleration has been constant since it began moving, at rest, a short time earlier at what time was the truck at the intersection?
2. An airplane travelling north at a constant speed is observed two miles due north of the Washington Monument at midnight and is five miles north six minutes later. Find the speed of the airplane, and the time it was right over the Washington Monument.
3. The area of one side of a cube is 9 square centimeters. What is its volume?

These are, of course, the *same three problems* as far as their *algebra* is concerned, but they all *mean* something. They all tell a little story in words that you must *visualize* and *transform into equations* so that you can use the rules of algebra to solve for a desired quantity.

It is impossible to give any *completely* general set of rules for transforming words into equations – often the right transformation is "domain specific" – it is a big part of what you learn when you learn physics, or chemistry, or economics. One tends to use certain symbols to stand for certain quantities, e.g. t for time in the first equation/problem above or A and V for area and volume respectively in the third, and if we fail to use the right symbol – for example the x we used instead of t in the *second* equation/problem, it is harder to see the correspondance. We're much more likely to interpret the equation for $y(x)$ as a line in a two dimensional $x - y$ plane with *no* time involved.

This leads to a bit of a paradox. If you learned your algebra mostly from word problems, you may have a hard time making sense out or knowing all of the operations you need to know, because it isn't always easy to come up with a word problem that requires you to use algebra to resolve relations like $x = z^{\pi}(z^{-1} + a)$ into an expression for a (until the day you encounter one in some discipline or other), so certain kinds of solution methodology are undertaught. Yet if you (more than likely) learned it mostly by merely *drilling* the operations in through your skull utterly devoid of meaning (quite possibly with indifferent success) you may have no idea how to interpret word problems as algebra and vice versa.

To do science you have to do *both*, and do them well!

In this review, however, we can't really do both at the same time – we have to pick one or the other to do *first*. For better or worse, we're going to start off by thinking about how to convert word problems into algebraic form and the *idea* of how to manipulate (transform) those forms using simple rules, since without these two concepts it is difficult to even discuss formal algebra. It will turn out that when we write down an equation, *no matter what* we've written down we can very likely find a circumstance where that very equation occurs and has *meaning* (as we demonstrated above). With this as motivation perhaps we'll be able to look at lots of equations and rules with a little more faith that there is some point to it all.

Here are a *few* rules that may help you in the process of *encoding a problem* in algebra: putting a word problem into the form of a set of (hopefully simple) algebraic relations one can then manipulate according to the rules of algebraic transformation to find a solution, and then translating the result (if necessary) back into words.

- Identify all the *quantities* given in the problem. Algebra doesn't work too well with qualities such as being "green" (unless greenness can somehow be converted into a numerical scale) but it can certainly deal with the "number of green balls in an urn containing 1000 balls". Common quantities that occur in problems include length, time, mass, temperature, count, an angle's size in degrees or radians, cost, age, area, volume (all of which carry *units*, sort of), and all sorts of pure (scalar, dimensionless) numbers like 4, π, e, $\sqrt{2}$. Ultimately, quantities are things that can be expressed as a *number*, which is why we started this book by talking about numbers. Well, duh!

- Assign a *symbol* for all of the quantities. Often it is a good idea to include quantities for which values are given as well as the unknowns. For example, we might let g stand for "number of green balls in the urn" and N stand for "the total number of balls of all colors in the urn" whose value in this particular problem *happens* to be 1000. Maybe N will end up being important in solving the problem; maybe not, but it is better to make it a symbol early. We can always substitute in any particular value such as $N = 1000$ *later* when it is time to do *arithmetic*.

- Sometimes you will have to *make symbols up* for quantities that are *not* given in a problem but that you know intuitively are important. Even if the total number of balls N isn't given, we know that if we're counting balls in an urn identified by color, we'll might well need symbols for each color the urn might contain as well as a symbol for the total. In fact, you should feel free to introduce new symbols at will as the problem develops, even temporary ones that you only keep for a few steps!

- Read the problem *carefully* to determine relevant *relationships*. This part is often very domain specific, as noted. For example, by talking about balls and urns, it seems likely that I'm about to talk about *probability* since drawing balls at random from an urn is a classic example used in many probability courses. In this case the relevant relationship might be $p = g/N$, the probability of drawing a green ball is the total number of green balls divided by the total number of balls of all colors.

In another context, it may be known that green balls have a mass of 20 grams, red balls have a mass of 10 grams, and blue balls have a mass of 30 grams, and you may be asked to find the maximum or minimum possible mass of all the balls in the urn. In a problem like this you might need all sorts of symbols and relations: (r and b and $r+g+b = N$ and $m_{\rm tot} = m_g * g + m_b * b + m_r * r$ and so on to lead you to conclude that $m_{\rm min} = m_g * g + m_r * (N - g)$, $m_{\rm max} = m_g * g + m_b * (N - g)$

- Once you have written down a symbolic representation of all the relationships you know that might be relevant to the problem, it is time to *think*. Visualize what's going ont. Identify the quantity (or quantities) you are trying to solve for. Then identify a strategy – a path for performing algebraic steps that isolate that quantity and allow you to evaluate it from known information substituted into the algebra.

- *Do not do arithmetic* in general until this very last step. If you reason with symbols, especially with "standard" symbols or symbols with implicit units, you can perform a number of checks on the result by just looking at the answer to see if it has the right units, varies the way you intuitively expect with the parameters of the problem. If you reduce the number of green balls and the probability of drawing a green ball goes *up*, there may be a problem! This also helps people who check your work – the algebra is *your reasoning*. Arithmetic is just a tedious but necessary step required to turn the reasoning into an actual answer.

Let me emphasize this latter point. Most *good* instructors will weight the reasoning far more strongly than the actual answer, especially in physics. I personally have been known – not even all that infrequently, actually – to give a student full credit for getting the right *algebraic* answer for a particularly difficult problem even if they then turn around and mispunch the numbers into their calculator when computing the final (numerical) answer. By the same token I've been known to give a student a *zero* on a problem where all they do is write down the exactly correct answer – with no work or exposition of their reasoning process whatsoever.

A correlary of this is that you should never try to "do algebra" with your calculator, by punching numbers into it in an order that you think untangles some expression hoping that you punch through to the right number. You can't check your work and any tiny error in keying will give you the wrong answer. Worse than wrong, as *nobody* – not you, not your instructor, not your grader – will be able to tell what you did wrong. Do algebra on *paper* by manipulating *symbols* through small steps that lead to your answer. You can then easily check, and so can your instructor!

If you practice the steps above in the context of whatever it is that you are studying, you will rapidly learn the right symbols to use to express things and become adept at transforming word problems into them. Once you have correctly reduced a problem to an algebraic formulation, however, it is usually still necessary to solve for an answer using the *steps* of algebra, the ways of transforming equalities (or inequalities) into *new* equalities that gradually isolate some desired variable as "an answer".

Let's review these rules for transformation.

3.1 Algebraic Transformations of Equations

The most common transformations of algebra applied to equalities (the most common case) are summarized below, with with examples. In all of the subsections below, we assume that we are operating on equations expressed in terms of a set of variables with symbols such as $a, b, c, \ldots x, y, z$. All the rules and transformation below will work for complex numbers, real numbers (as a subset of complex numbers), or integers (as a subset of real numbers). They will not all *necessarily* work for more complicated algebras where the variables can stand for e.g. matrices or quaternions.

We begin by discussing equality itself. Equality is a somewhat subtle quality in algebra.

3.1.1 Equality

In English, for a thing to be "equal" to another is to assert that two *different* things are in fact the *same* – a rather oxymoronic thing if one thinks about it. In mathematics it means that two (possibly) different *symbols* in fact stand for the *same thing* – a rather more sensible quality.

The simplest statement of equality, one that is *always* correct for *any* possible symbol that stands anything at all is the *tautology*:

$$a = a \tag{3.1}$$

That is, whatever a stands for is the same as itself. It is what it is. As a symbol for a number, it has whatever *value* it has. This law is so fundamental that it is difficult to imagine any sort of logical system where it is not true.

Simple or not, obvious or not, there are (amazingly enough) times we will want to start with this in a derivation. Again observe that in algebra the statement:

$$a = b \tag{3.2}$$

doesn't mean that a and b are different *things* that are the same. It means that a and b are two symbols that stand for the *same thing* – whereever the symbol a is used in any expression one can indifferently use b and end up with exactly the same thing.

While we are discussing basic equality, we will add to these fairly obvious statements the less obvious rule of *transitivity*. If

$$a = b \tag{3.3}$$

and

$$b = c \tag{3.4}$$

then

$$a = c \tag{3.5}$$

Again, this *seems* trivial, but it really is not. In mathematics, it means that all three symbols basically stand for the same thing. We'll use this rule implicitly all the time, since we will often change just one *side* of an equation with an algebraic rule applied to that side only, and the preservation of equality with the *other* side basically follows from transitivity. In any event it can't hurt to formally state the rule and name it at least one time.

3.2 Addition Rules

We are *not* going to precisely define the addition of numbers – we presume that the reader has at least learned to add real numbers somewhere by the time that they read this, and this skill also suffices to add complex numbers (where one simply adds the real and imaginary parts – all real numbers – separately).

Instead let us reiterate the *properties* of adding *symbols* for arbitrary numbers in whatever system we're working in.

$$a + (b + c) = (a + b) + c \quad \text{associativity} \tag{3.6}$$
$$a + b = b + a \quad \text{commutativity} \tag{3.7}$$
$$a + 0 = a \quad \text{identity} \tag{3.8}$$
$$a + (-a) = 0 \quad \text{inverse} \tag{3.9}$$

are all ways (however trivial) of changing one side or the other of an equation.

Note that we have introduced the notion of *parenthesizing* terms – grouping them together in an expression. You should think of parentheses as being *instructions* to the reader on the *order* that should be used when evaluating an algebraic expression. The rule is: Do the arithmetic (or algebra) inside parentheses *first*. If parentheses are nested, do the innermost ones first, then the next innermost, and so on out to the outermost. When all parenthesized expressions are evaluated, go ahead and add up everything else in left-right order (which will no longer matter because of commutativity).

For expressions containing addition *only* this won't matter because (for example – try it with any numbers you like):

$$1 + (3 + 4) = 1 + 7 = 8 = 4 + 4 = (1 + 3) + 4 = 1 + 3 + 4 \tag{3.10}$$

in any possible order or grouping but as we'll see, when we throw in multiplication it can matter a great deal!

Another addition rule that is extremely valuable that doesn't quite fit anywhere else is the following. Given two equations:

$$\begin{aligned} \{a &= b\} \\ +\{c &= d\} \\ (a + c) &= (b + d) \end{aligned} \tag{3.11}$$

In words, the sum of two equations (where we separately sum the *expressions* on the two sides of the equal signs) is an equation!

This rule is used a *lot* in algebra. Here are three common forms based on adding the same thing to both sides of an equation:

$$\begin{aligned} \{x &= y - b\} \\ +\{b &= b\} \\ x + b &= y - b + b \\ x + b &= y + (-b + b) \\ x + b &= y \end{aligned} \tag{3.12}$$

or $y = x + b$ (following a common but unnecessary convention of putting "the answer" on the left and the formula for obtaining it on the right).

Observe the full derivation of this rule. We added the tautology $b = b$ as an equation to another equation, getting an equation, then grouped and cancelled terms. We show *all* the steps (we usually won't) to emphasize precisely how we can build a new rule based on old ones we already know!

Note that subtraction from both sides is just adding a negative quantity and doesn't need a separate rule. That is:

$$\begin{aligned} \{x &= y + b\} \\ +\{-b &= -b\} \\ x + (-b) &= y + b + (-b) \\ x - b &= y + b - b \\ x - b &= y + (b - b) \\ x + b &= y \end{aligned} \tag{3.13}$$

When applying this rule to an equation as you try to solve for some quantity, it is easiest if you just visualize it as a *process*. Mentally *move any additive term* from one side of an equation to another while *changing its sign*:

$$\begin{aligned} x^2 &= y^3 - y + b \\ \rightarrow x^2 - b &= y^3 - y \end{aligned} \tag{3.14}$$

or

$$x^2 = y^3 - y + b$$

$$\begin{aligned} x^2 &= (y^3 - y) + b \\ \rightarrow x^2 - (y^3 - y) &= b \\ b &= x^2 - y^3 + y \end{aligned} \tag{3.15}$$

Note that by grouping one can move *whole sets of symbols at once* as long as they are *added* to one side or the other. We can move terms in any sum from the left to the right side of an equal sign or from right to left equally easily as in either case we are just adding a suitably framed tautology to the original equation!

A final extremely useful rule is to add *zero* to either side of an equation in a *symbolic form* such as $0 = a - a$. For example:

$$\begin{aligned} y &= z^2 \\ y &= z^2 + (a - a) \end{aligned} \tag{3.16}$$

This rule initially looks a bit silly. Why do we need it? We're adding something arbitrary to an equation that *cancels*, after all!

It turns out that this addition rule is critical to a process called *completing the square* that we'll use to derive the infamous *quadratic formula* later on.

To do algebra successfully, one needs to learn *all* of these rules so completely that one is never "remembering" them (as one does with things one has "memorized") but so that one *knows* them. You want to be able to use any of them *easily*, flipping terms from one side of an equation to another like lightning as easily as you breathe.

That require (more) examples and practice, practice, practice. But really, they are pretty easy to remember because if you think about them at all, they *make sense!*

Before we go on, since this is mathematics for *science* we should point out a *very important aspect* of using algebraic or numerical addition in e.g. physics, or chemistry, or mathematics. It has to do with *units*.

3.3 Multiplication Rules

Multiplication rules are very similar. Again we imagine that the reader knows how to multiply real numbers (using a calculator, even) so that the idea of multiplication of numbers as a *process* is familiar. In that case, the reader can easily verify for themselves that given real numbers a, b, c, the following are all true. They are true for *complex* a, b, and c as well (trust me), but that's a lot easier to show *after* we've worked out the rules for real numbers, in particular distributivity. We have to work a bit harder on inversion and division, as well.

$$a * (b * c) = (a * b) * c \quad \text{associativity} \tag{3.17}$$

$$a * b = b * a \quad \text{commutativity} \tag{3.18}$$

$$a * 1 = a \quad \text{identity} \tag{3.19}$$

$$a * \frac{1}{a} = a * a^{-1} = 1 \quad \text{inverse} \tag{3.20}$$

Once again we see that for expressions that contain only multiplied terms we can group them in any order:

$$1 * (3 * 4) = 1 * 12 = 12 = 3 * 4 = (1 * 3) * 4 = 1 * 3 * 4 \tag{3.21}$$

(again, try it for different numbers). No matter what order we multiply out (symbols for) numbers, we get the same result.

There are more rules

In physics one *can* multiply symbols with different units, such an equation with (net) units of meters times symbols given in seconds. In the end, however, the units on both sides must be consistent and make sense.

3.3.1 Dividing both sides of an equation by any scalar number or consistent symbol

Here one must be careful when performing symbolic divisions to avoid points where division is not permitted or defined (e.g. dividing by zero or a variable that might take on the value of zero). Note that dividing one unit by another in physics is also permitted, so that one can sensibly divide length in meters by time in seconds.

3.4 Order Convention for Mixed Forms

Parentheses *do* start to matter for mixed forms, expressions that contain both multiplication/division and addition/subtraction.

Suppose we look at the following equation:

$$y = a * x + b \tag{3.22}$$

We know that we can factor this somehow, but how we factor it depends a bit on how we *group terms* with *parentheses*. It could mean this:

$$y = a * (x + b) \tag{3.23}$$

or it could mean this:

$$y = (a * x) + b \tag{3.24}$$

These are *different*! Consider $a = 2$, $b = 3$, $c = 4$:

$$y = 2 * (3 + 4) = 14 \neq (2 * 3) + 4 = 10 \tag{3.25}$$

Hmmm, looks like *order matters!*

Parentheses can *always* be used to make the order of evaluation unambiguous, and if there is *any doubt* as to what the correct order should be they *should* be. However, algebracians and computer scientists (who often have to implement "algebra" in computer programs) don't want to *always* have to use them to resolve these conflicts. They (and we) introduce and *ordering convention* to tell us which operations to do first in an expression that mixes things like taking powers, multiplying and adding.

Starting at the deepest level of parentheses and working recursively outward:

1. Evaluate all powers

2. Evaluate all products

3. Evaluate all sums

where at any given level it *should not matter* if you work from left to right or right to left across the terms because both addition and multiplication are separately both *commutative* and *associative*.

Here's an example of an expression both with and without the (redundant) parentheses:

$$a * x * y^n + b * y + c = (a * x * (y^n)) + (b * y) + c \qquad (3.26)$$

First evaluate y^n. Multiply it by a and x. Multiply b and y. Add the results together.

With this rule defined, we can actually shorten our expression *still further* by adding another convention and still have expressions be quite unambiguous. From now on, if we put *two symbols next to each other* with *no sign in between them*, they are assumed to be *multiplied* with an implicit $*$ operator. That is:

$$axy^n + by + c = a * x * y^n + b * y + c = (a * x * (y^n)) + (b * y) + c \qquad (3.27)$$

Note that the expression on the left is *much shorter* than the one on the right – shorter means faster to write, easier (once you get the hang of it) to read!

3.5 Distributivity

Now that we understand parentheses and order, we can learn a very important rule for *distributing a product* or *factoring a sum* called the rule of *distributivity*. The following rule is used in either direction – left to right to distribute, right to left to factor:

$$a(b + c) = ab + ac \quad \text{distributivity} \qquad (3.28)$$

Note well that we are no longer writing in an explicit $*$. Did you read this (correctly) as:

$$a * (b + c) = (a * b) + (a * c) \qquad (3.29)$$

where we used the conventions defined above to determine the implicit order and sign?

Our repertoire of algebraic transformations is almost complete, or at least almost complete *enough* that we can solve a whole lot of things with them as is or use them to quickly derive new rules below that aren't worth putting in our *essential* list even though they may well be very useful for certain problems.

3.6 Power Rules

Powers are very important in algebra. We begin by defining the *power* of a symbol by the count of the number of times that symbol is to be multiplied by itself, that is:

$$a^n = a * a * ... * a * a \quad (n \text{ times}) \tag{3.30}$$

where n (the power) is obviously some integer, since we don't really know what it would mean (yet) to multiply out anything but discrete numbers.

However, the rules for multiplication also include division as multiplication of inverses. We've already indicated that the inverse relationship can be written as a *negative* power, $1/a = a^{-1}$. From this we can see that:

$$a^{-n} = \frac{1}{a * a * ... * a * a} \quad (n \text{ times}) \tag{3.31}$$

makes sense, again for n any integer.

Now consider the following expression:

$$b = a^n = a * a * ... * a * a \quad (n \text{ times}) \tag{3.32}$$

Note first of all that if a is any (say real) number, b will definitely be a real number because the reals are closed under multiplication. It is less obvious, but still true, that if $b \geq 0$ there is *always* at least one number $a > 0$ such that this equation is satisfied (you'll have to take my word for it for now, although later on you'll come to understand this in more detail if you ever need to).

If $b < 0$, we have a bit of a problem. If n is odd, once again there will always be at least one $a < 0$ for which this equation is true because the product of an odd number of negative numbers is negative. If n is *even* and the real number b is *negative*, however, we are in *trouble*. That is because there is no *real* number a such that e.g.

$$-1 = a^2 \tag{3.33}$$

Squaring a real number produces a *non-negative real number* as a result!

This, of course, is why we invented *complex* numbers. They are *extended* from the reals so they contain numbers like the imaginary unit i defined by:

$$-1 = i^2 \tag{3.34}$$

With or without this complex extension, we'll find it *very useful* to define the following a that "solves" $b = a^n$:

$$a = b^{1/n} \tag{3.35}$$

A real solution a will *always exist* for n odd and any real b. A real solution a will *always exist* for n even and a real $b \geq 0$. *No* real solution a will exist for n even and real $b < 0$, which is then a restriction on the *domain* of fractional powers over the set of all real numbers. Finally, a solution will *always exist* for any b and any n if a (and/or b) are permitted to be *complex*.

With this definition of fractional powers, we can now define the following algebraic rules that work for powers. These rules are *very important* and *extremely useful*. We defer any discussion of allowed domains or real versus complex until the end. We begin with:

$$0^n = 0 \qquad (n > 0) \tag{3.36}$$

$$0^n = \frac{1}{0} = \text{undefined} \qquad (n \leq 0) \tag{3.37}$$

There exists a sense where the latter can stand for ∞, but it is by far the safest to just say it is undefined, since ∞ *in the sciences* is generally *not a number* although (sigh) mathematicians have lots of harmless fun pretending that it is. In fact, we'll talk about ∞ later on in this work so you really understand the way the concept should be *used* in almost all practical applications so that its use yields an unambiguous answer. Ultimately this means that powers of zero are zero, the inverse of (or division by) zero is meaningless.

With a "special" pair of rules for $a = 0$ in hand above, we will now insist that $a \neq 0$ in all the rules below and that (at first) m and n be *integers*:

$$a^0 = 1 \tag{3.38}$$

$$a^m a^n = a^{(m+n)} \tag{3.39}$$

$$a^m a^{-n} = a^{(m-n)} \tag{3.40}$$

$$(a^m)^n = a^{mn} \tag{3.41}$$

$$(a^{\frac{1}{m}})^n = a^{\frac{n}{m}} \tag{3.42}$$

This last rule is extremely interesting to us. It tells us that subject to the domain restrictions above that allow us to define $a^{1/n}$ in the first place,

a^r is a meaningful number for *any rational number r!* Let's summarize the domain rules one more time as they are critically important:

1. If real number $a > 0$ there is *always* a real number $b = a^r$.

2. If real numb er $a < 0$ and if the rational number r has an odd denominator *or* is an integer, there is *always* a real number $b = a^r$.

3. If $a \neq 0$ is complex, there is *always* a complex $b = a^r$ for any rational r.

Don't (please!) try to memorize these rules. Just study them until you understand them and can remember them as a part of that understanding.

All of the algebraic power rules listed above therefore hold for any rational numbers m and n and not just integers, subject to these domain rules. This leaves us with an open question. We have proven that for very general domain restrictions (usually we just say $a > 0$ and forget about all the rest) a^r exists for for any rational number r. What about a^x for any *real* number x?

From a computational point of view it hardly matters. Any number you can punch into a calculator or program into a computer is clearly OK, and one can get (in some sense) "as close as you like" to any *irrational* number. So to a physicist or scientist, we just plain don't care. *Mathematicians* are a bit pickier – does (say) a^π exist? Note that in this case there *is no denominator* so that we can write this out as an integer power of the *inverse* of an integer power of a number, where all we originally defined the notion of the "power" of a number as being a certain integer number of products of that number. Our original notion of power has morphed into something *quite new.*

Without proof, the answer is pretty much yes. As is not infrequently the case the "best" way to think of powers is to *define* powers by the rules above on the most general *complex* domain, then consider special cases of this definition that work for e.g. real $a > 0$.

In fact, we'll do *one more extension* and a bit of simplification as we no longer need to worry showing that rational exponents work. Let us note that $b = a^x$ where a, b and x are complex numbers (yes, the exponent too!)

always has one or more solutions. In that case:

$$a^0 = 1 \quad (3.43)$$
$$a^x a^y = a^{(x+y)} \quad (3.44)$$
$$(a^x)^y = a^{xy} \quad (3.45)$$

holds on the complex plane. We'll have to wait until later to define just what is meant by taking a number to the ith power and the inverse relation implied by these rules, our friend the *logarithm*.

In the meantime, let's see what transformations these rules enable us to perform on equations containing powers.

3.6.1 Taking both sides of an equation to any power

This means that if we know that $q = az^2$ that we *also* know that:

$$q^n = (az^2)^n$$

for any real number n where the parentheses are necessary. We will more about powers below – they have special rules for tranforming the latter term into $a^n z^{2n}$ (for example).

Knowing the "special" rules for powers is especially important if the original equation can take on negative or complex values or has any sort of domain restrictions. Nevertheless, this is a very powerful transformation and one uses it quite often in solving problems. This rule will usually work even if the quantities have units.

3.6.2 Power Rules

This isn't quite the same as the rule above. Suppose $y = 3x$. Then:

$$e^y = e^{3x}$$

where e is a *constant*, in particular the "natural exponential constant" (although it would work for $a^y = a^{3x}$ just as well).

In physics (or anything else where the symbols can have units and aren't just pure numbers), this is *not* generally true unless the arguments are dimensionless. In fact this is a specific example of a *general rule* that one cannot substitute equalities that carry units into any functional form that has a power-series expansion. On the other hand one *can* substitute in quantities that in some sense "look the same" that are *dimensionless*. This is easy to understand. Supposed I know that x is a length in meters. I *can* certainly write down its exponential: e^x. But does this make sense?

If I expand e^x in its well-known power series (summarized, derived, defined later):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$$

we see that it is in fact *nonsense!* How can I add meters to meters-squared to meters-cubed? What is a length plus an area plus a volume?

The only way this can make sense is if we write something like:

$$e^{x/\lambda} = 1 + x/\lambda + \frac{(x/\lambda)^2}{2!} + \frac{(x/\lambda)^3}{3!} \ldots$$

where λ is some *other* variable or constant with units of length. Then each term in the sum is dimensionless, and there is nothing wrong with setting $\lambda = 1$ *meter* if that corresponds to the physical length scale of the problem.

3.7 Consistency of Units

In science, one is actually almost never adding up pure numbers. One is usually adding up numbers of *things*, or representing *physical quantities* with numbers. We therefore must carefully note that all of the beautify algebraic rules above apply *only* when the objects being added are *the same kind of thing!*

This is the fundamental "apples and oranges" rule. If I have six apples and seven oranges and add them together, I don't have thirteen apples, and I don't have thirteen oranges. Perhaps I have thirteen fruit, but if I do I should have written, and thought of, this as adding 6 fruit to 7 fruit. This is even more apparent if I subtract – who knows that subtracting two oranges from 6 apples even *means*? Adding two anti-oranges to the six apples?

This means that when we write:

$$y = ax + b \qquad (3.46)$$

in (say) *physics*, y, ax (as a product) and b had better all *have the same units* and describe the same kind of physical quantity, or be *dimensionless* – pure numbers. This puts some obvious limitations on what you can do with algebraic transformation!

As we noted above, in most scientific disciplines, we use the same symbols over and over again to stand for different specific kinds of quantities that carry different kinds of units that are either implicit – assumed by convention if no units are given – or explicitly indicated in the problem. In physics, the units are usually the Standard International (SI) unit set, where e.g. a mass symbol such as m or M is in kilograms, a time symbol t, t_0, or t_1 is in seconds, lengths like x, y, or r are in meters.

This means that expressions like $x + x^2 + x^3$ (the sum of a length, an area, and a volume) or $t + x + m$ (the sum of a time, a length, a mass) are *completely meaningless!*. Requiring consistency of units is a powerful (and additional) *difference* between ordinary "math" algebra and algebra used in application to the real world!

This actually is a great boon to those seeking to solve problems! You can take it as a given that *physical laws are always dimensionally consistent*. If you try to formulate and solve a problem using them and end up with things on two sides of an equal sign with *different units* or a sum on *one* side of an equals sign containing terms with different units, you can be absolutely certain that your answer is *wrong* and furthermore, that *you* made a mistake doing the algebra! This means that you can go back to the algebra and check and see where you multiplied where you should have divided or the like, and possibly solve the problem *correctly*.

Of course this is impossible if you've been "doing algebra with your calculator" by multiplying and adding *numbers* in the problem as you go instead of using *symbols with implicit units* to solve it algebraically *first*. Which is why you should almost never, ever do such a silly thing.

Solving a problem with algebra involves a) formulating the problem in meaningful symbols with implicit or explicit units; b) solving for the desired quantity or quantities using algebra; c) *checking the units of the result* and fixing your work if they are inconsistent; d) *only then* substituting in any given numbers and obtaining a number answer *that has the right units* because you just checked, didn't you?

Even if you get the wrong answer, your answer is unlikely to be *crazy* if it has the right units. Answers with inconsistent units are just plain crazy – they could *never* be right.

3.7.1 Placing the two sides of any equality into *almost* any functional or algebraic form as if they are variables of that function

There are several things to be careful of here both in real problems in science and in abstract problems in mathematics. In science the usual warnings about units hold as in the previous example. It may *look* OK to say $x = y$ where x and y are both in meters and then to form $\sin(x) = \sin(y)$, but the latter has a power series expansion and is dimensional nonsense. In mathematics one *has* to worry about the *domain and range* (defined below when we talk about functions). Suppose I have the relation $y = 2+x^2$ where x is a real dimensionless expression, and I wish to take the $\cos^{-1}$ of both sides. Well, the *range* of cosine is only -1 to 1, and my function y is clearly strictly larger than 2.

3.7.2 Inequalities

The same general set of rules holds for inequalities *except* that if one multiples both sides of an inequality by a negative number (or perform any other transformation with a similar effect) one must *change the direction* of the inequality as well. That is, If $x < y$, then:

$$-x = -1 * x > -y = -1 * y$$

That's pretty much it. *Most* solutions to almost *any* algebraically formulated problem involve taking small steps selected from the list above. It's easy once you get the hang of it! In fact, you can actually learn to "visualize" algebra, moving this symbol or that from one side of an equation to the other with your eyes alone even before you put the step down on paper.

Mind you, the list above left several things undone! For one thing, we haven't begun to talk about or define a number of words that we *used* in our listing above – words like "function", "power", "term", "domain", "range".

Once again we were faced with a bootstrapping issue and chose to begin with the rules themselves even if the rules contained some "forward references" to some things we'd rather discuss *with* the rules (at least some of them) in hand.

Let's get started cleaning all this up. First we'll define some very common algebraic operations, such as forming powers of a symbol. Then we'll talk about a special class of equations called *functions* – maps from one symbol (or set of symbols) onto another. Functions often describe relationships between quantities, which in turn have *meaning*, so this is a good step to have underway.

3.8 Powers

Chapter 4

Functions

4.1 Complex Numbers and Harmonic Trigonometric Functions

Some *extremely useful and important* True Facts:

4.1.1 Complex Numbers

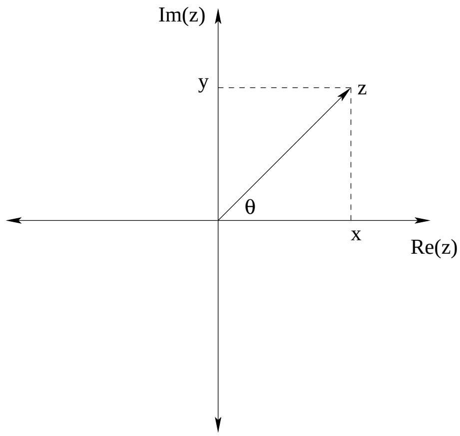
This is a very terse review of their most important properties. An arbitrary complex number z can be written as:

$$z = x + iy \tag{4.1}$$
$$= |z|\cos(\theta) + i|z|\sin(\theta) \tag{4.2}$$
$$= |z|e^{i\theta} \tag{4.3}$$

where $x = |z|\cos(\theta)$, $y = |z|\sin(\theta)$, and $|z| = \sqrt{x^2 + y^2}$. *All complex numbers* can be written as a real amplitude $|z|$ times a complex exponential form involving a phase angle. Again, it is difficult to convey how incredibly useful this result is without further study, but I commend it to your attention.

There are a number of really interesting properties that follow from the exponential form. For example, consider multiplying two complex numbers a and b:

$$a = |a|e^{i\theta_a} = |a|\cos(\theta_a) + i|a|\sin(\theta_a) \tag{4.4}$$



$$b = |b|e^{i\theta_b} = |b|\cos(\theta_b) + i|b|\sin(\theta_b) \tag{4.5}$$
$$ab = |a||b|e^{i(\theta_a+\theta_b)} \tag{4.6}$$

and we see that multiplying two complex numbers multiplies their *amplitudes* and *adds* their phase angles. Complex multiplication thus *rotates and rescales* numbers in the complex plane.

4.1.2 Trigonometric and Exponential Relations

$$e^{\pm i\theta} = \cos(\theta) \pm i\sin(\theta) \tag{4.7}$$
$$\cos(\theta) = \frac{1}{2}\left(e^{+i\theta} + e^{-i\theta}\right) \tag{4.8}$$
$$\sin(\theta) = \frac{1}{2i}\left(e^{+i\theta} - e^{-i\theta}\right) \tag{4.9}$$

From these relations and the properties of exponential multiplication you can painlessly prove all sorts of trigonometric identities that were immensely painful to prove back in high school

4.1.3 Power Series Expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \tag{4.10}$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \tag{4.11}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \tag{4.12}$$

Depending on where you start, these can be used to prove the relations above. They are most useful for getting expansions for small values of their parameters. For small x (to leading order):

$$e^x \approx 1 + x \tag{4.13}$$

$$\cos(x) \approx 1 - \frac{x^2}{2!} \tag{4.14}$$

$$\sin(x) \approx x \tag{4.15}$$

$$\tan(x) \approx x \tag{4.16}$$

We will use these fairly often in this course, so learn them.

4.1.4 An Important Relation

A relation I will state without proof that is very important to this course is that the real part of the $x(t)$ derived above:

$$\Re(x(t)) = \Re(x_{0+}e^{+i\omega t} + x_{0-}e^{-i\omega t}) \tag{4.17}$$

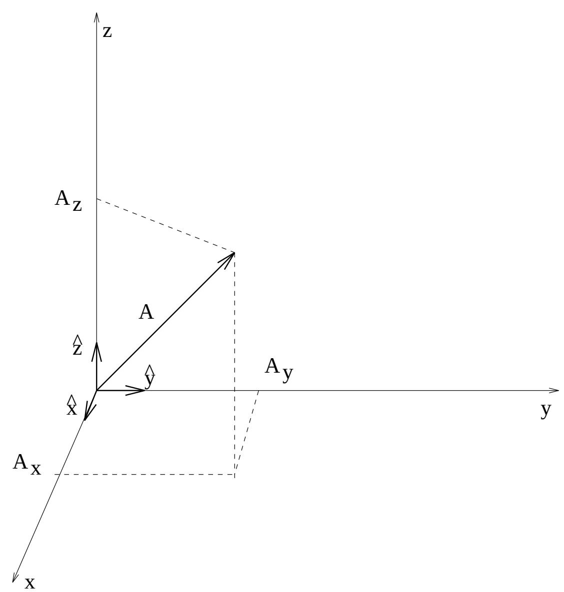
$$= X_0 \cos(\omega t + \phi) \tag{4.18}$$

where ϕ is an arbitrary phase. You can prove this in a few minutes or relaxing, enjoyable algebra from the relations outlined above – remember that x_{0+} and x_{0-} are arbitrary *complex* numbers and so can be written in complex exponential form!

Chapter 5

Coordinate Systems, Points, Vectors

5.1 Review of Vectors



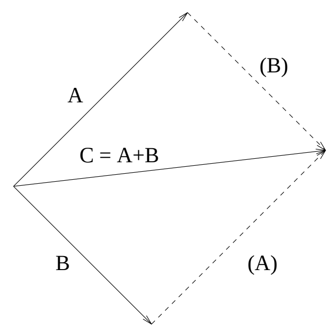
(5.1)

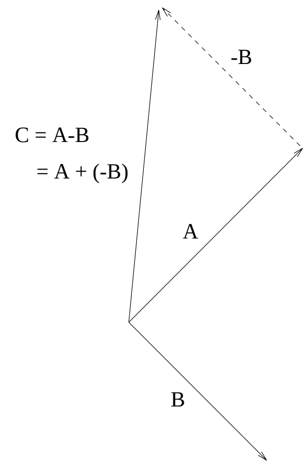
Most motion is not along a straight line. If fact, almost no motion is along a line. We therefore need to be able to describe motion along *multiple dimensions* (usually 2 or 3). That is, we need to be able to consider and evaluate *vector* trajectories, velocities, and accelerations. To do this, we must first learn about what vectors are, how to add, subtract or decompose a given vector in its cartesian coordinates (or equivalently how to convert between the cartesian, polar/cylindrical, and spherical coordinate systems), and what scalars are. We will also learn a couple of products that can be constructed from vectors.

A bf vector in a coordinate system is a directed line between two points. It has **magnitude** and **direction**. Once we define a coordinate origin, each particle in a system has a **position vector** (e.g. – $\vec{A}$) associated with its location in space drawn from the origin to the physical coordinates of the particle (e.g. – (A_x, A_y, A_z)):

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \tag{5.2}$$

5.1.1 Coordinate Systems and Vectors





The position vectors clearly depend on the choice of coordinate origin. However, the **difference vector** or **displacement vector** between two position vectors does **not** depend on the coordinate origin. To see this, let us consider the **addition** of two vectors:

$$\vec{A} + \vec{B} = \vec{C} \tag{5.3}$$

Note that vector addition proceeds by putting the tail of one at the head of the other, and constructing the vector that completes the triangle. To numerically evaluate the sum of two vectors, we determine their components and add them componentwise, and then reconstruct the total vector:

$$C_x = A_x + B_x \tag{5.4}$$

$$C_y = A_y + B_y \tag{5.5}$$

$$C_z = A_z + B_z \tag{5.6}$$

If we are given a vector in terms of its **length** (magnitude) and **orientation** (direction angle(s)) then we must evaluate its cartesian components before we can add them (for example, in 2D):

$$A_x = \left|\vec{A}\right| \cos(\theta_A) \qquad B_x = \left|\vec{B}\right| \cos\theta_B \tag{5.7}$$

$$A_y = \left|\vec{A}\right| \sin(\theta_A) \qquad B_y = \left|\vec{B}\right| \sin\theta_B \tag{5.8}$$

This process is called **decomposing** the vector into its cartesian components.

The **difference** between two vectors is defined by the addition law. Subtraction is just adding the negative of the vector in question, that is, the vector with the **same** magnitude but the **opposite** direction. This is consistent with the notion of adding or subtracting its components. Note well: Although the vectors themselves may depend upon coordinate system, the difference between two vectors (also called the **displacement** if the two vectors are, for example, the postion vectors of some particle evaluated at two different times) does **not**.

When we reconstruct a vector from its components, we are just using the law of vector addition itself, by **scaling** some special vectors called **unit vectors** and then adding them. Unit vectors are (typically perpendicular) vectors that define the essential directions and orientations of a coordinate system and have unit length. Scaling them involves multiplying these unit vectors by a number that represents the magnitude of the vector component. This scaling number has no direction and is called a **scalar**. Note that the product of a vector and a scalar is always a vector:

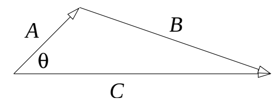
$$\vec{B} = C\vec{A} \tag{5.9}$$

where C is a scalar (number) and $\vec{A}$ is a vector. In this case, $\vec{A}||\vec{B}$.

Finally, we aside from multiplying a scalar and a vector together, we can define products that multiply two vectors together. By "multiply" we mean that if we double the magnitude of either vector, we double the resulting product – the product is *proportional* to the magnitude of either vector. There are two such products for the ordinary vectors we use in this course, and both play *extremely important roles* in physics.

The first product creates a scalar (ordinary number with magnitude but no direction) out of two vectors and is therefore called a **scalar product** or (because of the multiplication symbol chosen) a **dot product**. A scalar is often thought of as being a "length" (magnitude) on a single line. Multiplying two scalars on that line creates a number that has the *units* of length squared but is geometrically not an area. By selecting as a direction for that line the direction of the vector itself, we can use the scalar product to *define* the length of a vector as the *square root* of the vector magnitude times itself:

$$\left|\vec{A}\right| = +\sqrt{\vec{A} \cdot \vec{A}} \tag{5.10}$$



From this usage it is clear that a scalar product of two vectors can never be thought of as an area. If we generalize this idea (preserving the need for our product to be symmetrically proportional to both vectors, we obtain the following definition for the general scalar product:

$$\vec{A} \cdot \vec{B} = A_x * B_x + A_y * B_y \ldots \quad (5.11)$$

$$= \left|\vec{A}\right| \left|\vec{B}\right| \cos(\theta_{AB}) \quad (5.12)$$

This definition can be put into *words* – a scalar product is the length of one vector (either one, say $|\vec{A}|$) times the *component* of the other vector ($|\vec{B}| \cos(\theta_{AB})$ that points in the *same direction* as the vector $\vec{A}$. Alternatively it is the length $|\vec{B}|$ times the component of $\vec{A}$ parallel to $\vec{B}$, $|\vec{A}| \cos(\theta_{AB})$. This product is *symmetric* and *commutative* ($\vec{A}$ and $\vec{B}$ can appear in either order or role).

The other product multiplies two vectors in a way that creates a third vector. It is called a **vector product** or (because of the multiplication symbol chosen) a **cross product**. Because a vector has magnitude and direction, we have to specify the product in such a way that both are defined, which makes the cross product more complicated than the dot product.

As far as magnitude is concerned, we already used the non-areal combination of vectors in the scalar product, so what is left is the product of two vectors that makes an *area* and not just a "scalar length squared". The area of the parallelogram defined by two vectors is just:

$$\text{Area in } \vec{\mathrm{A}} \times \vec{\mathrm{B}} \text{ parallelogram} = \left|\vec{A}\right| \left|\vec{B}\right| sin(\theta_{AB}) \quad (5.13)$$

which we can interpret as "the magnitude of $\vec{A}$ times the component of $\vec{B}$ perpendicular to $\vec{A}$" or vice versa. Let us accept this as the magnitude of the cross product (since it clearly has the proportional property required) and look at the direction.

The area is nonzero only if the two vectors do *not* point along the same line. Since two non-colinear vectors always lie in (or define) a plane (in which the area of the parallelogram itself lies), and since we want the resulting product to be independent of the coordinate system used, one sensible direction available for the product is along the line *perpendicular to this plane*. This still leaves us with *two* possible directions, though, as the plane has two sides. We have to pick one of the two possibilities by *convention* so that we can communicate with people far away, who might otherwise use a counterclockwise convention to build screws when we used a clockwise convention to order them, whereupon they send us left handed screws for our right handed holes and everybody gets all irritated and everything.

We therefore *define* the direction of the cross product using the *right hand rule*:

> Let the fingers of your *right hand* lie along the direction of the first vector in a cross product (say $\vec{A}$ below). Let them curl naturally through the *small angle* (observe that there are two, one of which is larger than π and one of which is less than π) into the direction of $\vec{B}$. The erect *thumb* of your right hand then points in the general direction of the cross product vector – it at least indicates which of the two perpendicular lines should be used as a direction, unless your thumb and fingers are all double jointed or your bones are missing or you used your left-handed right hand or something.

Putting this all together mathematically, one can show that the following are two equivalent ways to write the cross product of two three dimensional vectors. In components:

$$\vec{A}\times\vec{B} = (A_x{*}B_y - A_y{*}B_x)\hat{z} + (A_y{*}B_z - A_z{*}B_y)\hat{x} + (A_z{*}B_x - A_x{*}B_z)\hat{y} \quad (5.14)$$

where you should note that x, y, z appear in *cyclic order* (xyz, yzx, zxy) in the positive terms and have a minus sign when the order is *anticyclic* (zyx, yxz, xzy). The product is *antisymmetric* and *non-commutative*. In particular

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A} \quad (5.15)$$

or the product *changes sign* when the order of the vectors is reversed.

Alternatively in *many* problems it is easier to just use the form:

$$\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right| sin(\theta_{AB}) \quad (5.16)$$

to compute the magnitude and assign the direction *literally* by (right) "hand", along the right-handed normal to the AB plane according to the right-hand rule above.

Note that this *axial* property of cross products is realized in nature by things that *twist* or *rotate around an axis.* A screw advances into wood when twisted clockwise, and comes out of wood when twisted counterclockwise. If you let the fingers of your right hand curl around the screw *in the direction of the twist* your *thumb* points in the direction the screw moves, whether it is in or out of the wood. Screws are therefore by convention *right handed.*

One final remark before leaving vector products. We noted above that scalar products and vector products are closely connected to the notions of *length* and *area*, but mathematics per se need not specify the *units* of the quantities multiplied in a product (that is the province of physics, as we shall see). We have numerous examples where two *different* kinds of vectors (with different units but referred to a common coordinate system for direction) are multiplied together with one or the other of these products. In actual fact, there often *is* a buried squared length or area (which we now agree are different kinds of numbers) in those products, but it won't always be obvious in the dimensions of the result.

Two of the most important uses of the scalar and vector product are to define the *work* done as the force through a distance (using a scalar product as work is a scalar quantity) and the *torque* exerted by a force applied at some distance from a center of rotation (using a vector product as torque is an axial vector). These two quantities (work and torque) have the *same units* and yet are very *different* kinds of things. This is just one example of the ways geometry, algebra, and units all get mixed together in physics.

At first this will be very confusing, but remember, back when you where in third grade multiplying *integer numbers* was very confusing and yet rational numbers, irrational numbers, general real numbers, and even complex numbers were all waiting in the wings. This is more of the same, but all of the additions will *mean something* and have a compelling *beauty* that comes out as you study them. Eventually it all makes very, very good sense.

5.2 Calculus

5.2.1 Power Law Integration and differentiation

$$f = Ax^n \tag{5.17}$$

$$\frac{df}{dx} = nAx^{n-1} \tag{5.18}$$

$$\int f dx = \frac{1}{n+1}Ax^{n+1} + C \tag{5.19}$$