Simple rate-equation model for two-photon lasers

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We present a rate-equation model for two-photon lasers that, despite its simplicity, captures the essential physics of their behavior and affords an intuitive understanding of their novel threshold and stability behavior. We use the model to investigate the steady-state behavior of the laser, explore the stability of the steady-state solutions, and predict the injected pulse strength needed to initiate lasing.

Modeling the behavior of two-photon lasers is challenging because such lasers are based on the two-photon stimulated-emission (SE) process and hence operate in a highly nonlinear manner under all conditions. Recall that in the two-photon SE process two incident photons stimulate an inverted atom to a lower energy state, and four photons are scattered coherently by the atom. We have developed a model for two-photon lasers based on a set of self-consistent rate equations that predict many of their crucial attributes without being overly complex. In this Letter we present our rate-equation model and use it to make predictions of the behavior of two-photon optical lasers. Although more comprehensive treatments that include interactions ignored in our rate-equation model, such as coherent effects, single-photon processes, and dynamical Stark shifts, may be needed to make quantitative comparisons with experimental results, we feel that our model is useful for developing an intuitive understanding of two-photon lasers.

A key test for any model of two photon lasers is whether it predicts the novel threshold behavior of the laser that we briefly describe below. The threshold condition for all lasers is that the round-trip gain must equal the round-trip loss. For one-photon lasers this criterion yields the well-known result that lasing will commence when a uniquely defined minimum inversion (proportional to the gain) is attained by means of sufficient pumping. The situation is more complicated for the two-photon laser because the gain increases with increasing inversion \( AN \) and with increasing cavity photon number \( q \) (until the atoms are saturated); thus the threshold condition must be specified by two parameters. We define a threshold inversion \( AN^{th} \) as the inversion needed to satisfy the threshold condition with cavity photon number \( q_{sat} \) just sufficient to saturate the two-photon gain. When \( AN > AN^{th} \) there is a corresponding cavity photon number (which is less than \( q_{sat} \)) that must be present in the cavity before the laser can be turned on. Hence, if the laser is initially off, it cannot be turned on unless some perturbation, such as an externally injected field, brings it above the necessary value.

Our rate-equation model of the two-photon laser follows from the standard model of one-photon lasers with the exception that the one-photon SE rate

\[ W^{(1)} = B^{(1)} q \]

is replaced by the two-photon SE rate

\[ W^{(2)} = B^{(2)} q^2, \]

where \( B^{(1)} \) [\( B^{(2)} \)] is the one- (two-) photon rate coefficient. For simplicity we have assumed that the two-photon laser operates in the degenerate mode, that the laser oscillates in a single plane-wave mode, and that the cavity (population) decay rate \( \gamma_c(\gamma) \) is much smaller than the atomic coherence dephasing rate. Under these oversimplifying conditions the behavior of the laser is described by the mean photon number \( q \) in the cavity and the mean population inversion \( AN \) between the atomic levels that participate in the SE process. The first-order coupled nonlinear differential equations governing the evolution of these quantities are given by

\[ \frac{dq}{dt} = V_c B^{(2)} q^2 AN - \gamma_c[q - q_{inj}(t)], \]

\[ \frac{dAN}{dt} = -2B^{(2)} q^2 AN - \gamma(AN - AN_0), \]

where \( V_c \) is the volume of the gain medium within the cavity mode, \( AN_0 \) is the inversion in the absence of the field that is due to the pump process, \( \gamma AN_0 \) is the pump rate, and \( q_{inj}(t) \) is the photon number injected into the cavity by an external source. We see from Eq. (3) that the photon number increases as a result of the two-photon SE process and the injection from the external source, and it decreases as a result of linear loss through the cavity mirrors. We have ignored the possibility of two-photon spontaneous-emission processes at the laser frequency because the emission rates are extremely small in the optical regime. This approximation is not valid for two-photon masers, for which the stimulated and spontaneous rates are comparable. From Eq. (4) we see that the inversion decreases in response to the SE process and as a result of other radiative (at frequencies distinct from the laser frequency) and nonradiative decay mechanisms, and it increases in response to the pump process.
The steady-state behavior of the two-photon laser can be obtained readily from our model. From Eq. (4) we find that

$$\Delta N_s = \frac{\Delta N_0}{1 + q_{ss}^2/q_{sat}^2},$$

where $q_{sat} = \sqrt{\gamma/2E_{th}}$ is the standard definition of the two-photon saturation photon number. Equation (5) is reminiscent of the steady-state inversion for a one-photon laser,\textsuperscript{13} except that the denominator is not linear in $q_{ss}$. In contrast, the steady-state solution for the photon number is very different from that of one-photon lasers because it can be multivalued. We find three solutions, given by

$$q_{ss}^0 = 0,$$

$$q_{ss}^\pm = \frac{q_{imj}}{4\gamma_c} \sqrt{\Delta N_0 \pm (\Delta N_0^2 - 16q_{sat}^2\gamma_c^2/V_a^2\gamma^2)^{1/2}},$$

for the case when $q_{inj}(t) = 0$. The physically meaningful (real) steady-state solutions represented by these equations are shown in Fig. 1, where they are plotted as a function of the pump rate. It is seen from Eq. (7) that $\Delta N_0^\text{th} = 4q_{sat}\gamma_c/\gamma V_a^2$ is the minimum value of $\Delta N_0$ that admits of a nonzero photon number. This yields $\Delta N_s = (1/2)\Delta N_0^\text{th}$ and $q_{ss}^\pm = q_{sat}$ at threshold, in agreement with the heuristic discussion of the threshold behavior presented above.

The discontinuous threshold behavior shown in Fig. 1 is indicative of a first-order phase transition, which is different from the smooth turn-on behavior of normal one-photon lasers. Note from Fig. 1(a) that the two-photon gain is saturated at threshold, again in sharp contrast to the typical one-photon laser that operates far below saturation. Figure 1(b) also shows that the inversion is never constant, unlike the behavior of one-photon lasers for which the inversion clamps above threshold.\textsuperscript{13} Rate-equation models that assume the inversion remains constant were investigated previously.\textsuperscript{9}

Based on past experience with one-photon lasers we suspect that the steady-state solutions may be unstable because the laser operates in the saturated regime. We have performed a linear stability analysis of the steady-state solutions and find that (1) the zero-photon solution $(q_{ss}^0, \Delta N_0^\text{th})$ is always stable (two-photon spontaneous emission, neglected here, can destabilize this solution); (2) the $(q_{ss}^\pm, \Delta N_0^\text{th})$ solution, in which the photon number decreases with increasing pump rate, is always unstable; (3) the $(q_{ss}^\pm, \Delta N_0^\text{th})$ solution is always stable for a good cavity $(\gamma/\gamma_c > 1)$; and (4) the $(q_{ss}^0, \Delta N_0^\text{th})$ solution is unstable for a bad cavity $(\gamma/\gamma_c < 1)$ for pumping just above threshold but stabilizes for higher pump rates.\textsuperscript{4,6}

Our result (4) is inconsistent with the work of Heatley et al.\textsuperscript{14} and Ning and Haken,\textsuperscript{3} who found that there is no stable lasing state in the bad-cavity limit. We attribute this difference to our neglect of coherent effects based on the work of Ref. 3.

To address how an injected field can turn on the laser we have determined the steady-state solutions of Eqs. (3) and (4) and their stability properties when a continuous-wave beam of photons is injected into the cavity $(q_{inj} \neq 0)$. The stability of the solutions can be quite complex, especially in the bad-cavity limit. However, the analysis is straightforward if we consider the important case when the laser is initially off, $\Delta N_0 > \Delta N_0^\text{th}$, and $q_{inj}$ is increased slowly. This behavior is illustrated in Fig. 2(a), where we plot the three solutions for the steady-state photon number when $q_{inj} \neq 0$ and $\Delta N_0 = 1.2\Delta N_0^\text{th}$. The low-power solution (dashed curve) increases as $q_{inj}$ increases and is stable until it reaches a critical value $q_{inj}^\text{th} = 0.11q_{sat}$. This point defines the minimum injected photon number necessary to initiate lasing. As $q_{inj}$ increases beyond $q_{inj}^\text{th}$, the laser is forced to switch from the low-power solution (now unstable) to the high-power solution (stable), and the laser turns on. The behavior of the laser as $q_{inj}$ decreases depends on the quality of the cavity. For a good cavity the laser will continue to operate at high power as $q_{inj}$ decreases. For a bad cavity the laser will continue to operate at high power only if the solution is stable when $q_{inj} = 0$. We have studied the stability behavior for other values of the pump rate and find that the injection threshold $q_{inj}^\text{th}$ decreases for higher values of the pump rate, as shown in Fig. 2(b). In practice, a pulse (peak photon number $q_{p0}$) rather than a cw beam is injected into the laser. When the laser can (cannot) adiabatically follow the temporal variation of the pulse, $q_{inj} = q_{inj}^\text{th}$ $(q_{inj}^0 \geq q_{inj}^\text{th})$.\textsuperscript{3,4,6}
The transient behavior of the laser described above is explored by numerical integration of Eqs. (3) and (4). Figure 3 shows how the laser responds to injected trigger pulses for a good cavity ($\gamma_o/\gamma_c = 2$) when the pump rate is greater than the threshold pump rate ($\Delta N_o/\Delta N_{th} = 1.2$) and when there are no photons in the cavity initially. For a weak trigger pulse [peak amplitude $q_{inj}^0 = 0.1q_{sat}$; Fig. 3(a)] the laser is not driven above threshold, whereas for a slightly stronger pulse [peak amplitude $q_{inj}^0 = 0.12q_{sat}$; Fig. 3(b)] the laser is driven above threshold, and it attains a constant amplitude after the injected pulse is switched off. This is in good agreement with the injection threshold $q_{th} = 0.11q_{sat}$ calculated above.

The transient behavior shown in the plots of Fig. 3 is reminiscent of the experimental data on the dressed-state two-photon laser, with the exception that the rate-equation model does not predict the spiking during the initial turn-on of the laser nor the oscillatory behavior of the laser. It is known that models incorporating coherent effects lead to stable oscillating solutions, although it is not clear whether these effects alone properly account for the observed behavior. The observed spiking behavior during turn-on is an open question because no previous research has specifically investigated pulsed injection of a continuous-wave two-photon laser. However, we suspect that the behavior can be attributed to ac Stark shifts and coherent effects.

The response of the laser is more complicated for the case of a bad cavity ($\gamma_o/\gamma_c = 0.2$). As an illustrative example, we consider how the stability of the ($q_{sat}^*, \Delta N_{th}$) solution changes with pump rate. For pump rates just above threshold ($1 < \Delta N_o/\Delta N_{th} < 1.25$) the high-power solution is unstable, and hence an injected pulse cannot turn on the laser. For higher pump rates the solution becomes stable, large spiking occurs in the initial turn-on, and for $1.25 < \Delta N_o/\Delta N_{th} < 2.3$, the laser displays damped oscillatory behavior as it approaches steady state.

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References

1. There has been a great deal of research investigating two-photon lasers over the past 30 years. A good working list of references can be found in Ref. 2.