Problem 1 [10 pts]

A semi-detached binary star system is composed of a White Dwarf primary of mass $M_1 = 0.85 M_\odot$ and radius $R_1 = 0.0095 R_\odot$ and a secondary star of mass $M_2 = 0.17 M_\odot$. The orbital period of the system is $T = 0.0745\,\text{days}$.
Sketch what this system looks like, clearly showing and calculating:

- the separation between the centres of the two stars (both in units of m **and** $R_\odot$)
- the distances between both stars and the inner Langragian point L_1
- an estimate for the radius R_2 of the secondary star

Problem 2 [20 pts]

In class we obtained the following differential equation that we need to solve to get the scale factor as a function of time, $R(t)$:

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right] R^2 = -kc^2$$

where R and ρ are both functions of t.

(a) Show that we can rewrite this in terms of R and t only thus:

$$\left(\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho_0\frac{1}{R} = -kc^2$$

where ρ_0 is a constant, the average density of the present universe.

(b) From this equation, verify the following solutions for $R(t)$ for a flat, closed, and open universe:

$$R(t) = \left(\frac{3}{2}\right)^{2/3}\left(\frac{t}{t_H}\right)^{2/3} \qquad \text{(for k = 0)}$$

$$R(t) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1-\cos x) \quad \text{where,} \quad t = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}(x - \sin x) \qquad \text{(for k > 0)}$$

$$R(t) = \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}(\cosh x - 1) \quad \text{where,} \quad t = \frac{1}{2H_0}\frac{\Omega_0}{(1-\Omega_0)^{3/2}}(\sinh x - x) \qquad \text{(for k < 0)}$$

For the flat universe case explicitly solve the differential equation for R above, but for the closed and open universe cases it is sufficient to simply substitute these solutions into the differential equation to show they do in fact solve it.

(c) Sketch the form of these solutions on a plot of $R(t)$ versus t.

(d) Show that from these solutions we can obtain the time(age) of the universe as function of the redshift z thus (this is a bit messy, sorry!):

$$\frac{t(z)}{t_H} = \frac{2}{3}\frac{1}{(1+z)^{3/2}} \qquad \text{(for k = 0)}$$

$$\frac{t(z)}{t_H} = \frac{\Omega_0}{2(\Omega_0-1)^{3/2}}\left[\cos^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0}\right) - \frac{2\sqrt{(\Omega_0-1)(\Omega_0 z+1)}}{\Omega_0(1+z)}\right] \qquad \text{(for k > 0)}$$

$$\frac{t(z)}{t_H} = \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\left[-\cosh^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0 z + \Omega_0}\right) + \frac{2\sqrt{(1-\Omega_0)(\Omega_0 z+1)}}{\Omega_0(1+z)}\right] \qquad \text{(for k < 0)}$$

where Ω_0 is the density parameter of the present universe, as defined in class.

(e) For density parameters of $\Omega_0 = 1$, $\Omega_0 = 2$, and $\Omega_0 = 0.5$, calculate the age of the universe for each as ratio of t_H ($t_H = 13.6$ billion years for $h = 0.72$).