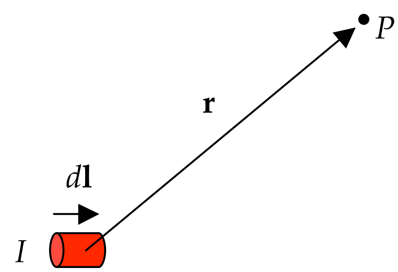
Physics 54

Sources of the Magnetic Field

Caution: Cape does not enable user to fly.
— Label on Batman Costume

The Biot-Savart Law

Soon after they learned of Oersted's discovery that a current produces magnetic effects, Biot and Savart undertook careful experiments to determine the details. In modern notation and terminology, their finding concerns the B-field set up by infinitesimal segment of a current-carrying wire.



We are interested in the B-field at point P due to the infinitesimal bit of wire shown, with current in the direction specified by the vector $d\mathbf{l}$.

The experimental answer is a fundamental law:

Biot-Savart Law	$d\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{d\mathbf{l} \times \mathbf{r}}{r^3}$

In this formula we have introduced another universal constant, $\mu_0 = 4\pi \times 10^{-7}$ in SI units (exactly, by definition).

The choice of this constant makes the *ampere* the defining quantity for SI electromagnetic units.

One can also write $\mathbf{r}/r^3 = \hat{\mathbf{r}}/r^2$, where $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$. This shows that the strength of the field falls off with distance from its source as $1/r^2$, just as in the electric case.

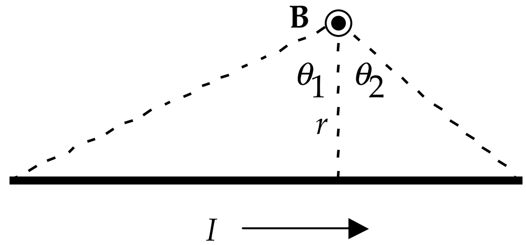
Of course there are never isolated infinitesimal bits of wire; one must integrate over all the current segments in the system in order to find the total B-field at P. In general such an integration is complicated; we will do it only in a few very simple cases. Nevertheless, the Biot-Savart law gives us the magnetic equivalent of the E-field of a point charge. In principle, the B-field of any set of currents could be calculated using it.

Two Examples

The simplest case geometrically is that of a straight wire carrying current I. This is also unphysical by itself, because steady currents cannot start and stop at the ends of a finite piece of wire. But circuits of rectangular shape are made of straight pieces, and the B-field of such a circuit can be obtained by adding their separate contributions.

The situation is shown. The integral over the wire is not difficult, and the answer is

$$B = \frac{\mu_0}{4\pi}\frac{I}{r}(\sin\theta_1 + \sin\theta_2)$$



The direction of **B** is out of the page, as indicated. If the field point were below the wire, the direction would be into the page. *The lines of the B-field form circles concentric with the wire.* There is another right-hand rule for this:

> **Point the thumb of the right hand in the direction of the current; the B-field lines curl like the fingers.**

This suggests that the lines of the B-field (unlike those of the electrostatic field) form closed curves. We will see that this is always true.

For points close to the wire and not close to one end, both angles in the above formula approach $\pi/2$, and we have a useful approximate formula:

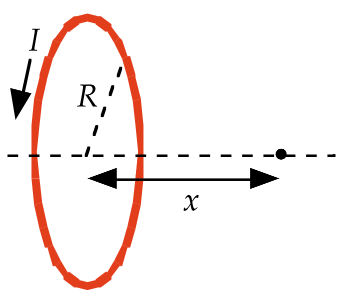
$$B = \frac{\mu_0}{2\pi}\frac{I}{r}$$

This would hold exactly, of course, for the obviously unphysical case of an infinite wire. We will use it often as an approximation.

Another simple geometry is that of a circular loop of wire carrying a current. This is an important case. It is easy to find the field at a point on the symmetry axis as shown — but not at all easy to find it at points off that axis.

Integrating the Biot-Savart law around the loop, we find for the B-field at the indicated point:

$$B = \frac{\mu_0 I}{2}\frac{R^2}{(R^2 + x^2)^{3/2}}.$$



The direction of **B** is parallel to the axis. For the current as indicated, it is to the right. Another right-hand rule governs this direction:

> **Curl the fingers of the right hand they way the current goes around the loop; the B-field on the axis is in the direction of the thumb.**

Far from the loop ($x >> R$) we have approximately

$$B \approx \frac{\mu_0 I R^2}{2x^3} = \frac{\mu_0}{2\pi}\frac{\mu}{x^3}$$

Here μ is the magnitude of the magnetic dipole moment of the loop, equal to the current times the area πR^2. (This symbol μ here is not to be confused with μ_0.) As in the electric case, the B-field of a dipole moment falls off with distance as the inverse cube.

General Properties of the Magnetic Field

The Biot-Savart law is the magnetic counterpart of Coulomb's law. From it one can derive two important general properties of the B-field.

The first concerns the flux through a closed surface. We define magnetic flux the same way we defined electric flux, just replacing **E** by **B**. Then we find a magnetic version of Gauss's law:

Magnetic Gauss's Law	$\oint \mathbf{B} \cdot d\mathbf{A} = 0$

The right side is zero because there are no isolated magnetic poles, so the *total* magnetic "charge" enclosed in a surface is always zero. In terms of field lines, this says:

Lines of the B-field always close on themselves.

The other property concerns the line integral of B along a closed path:

Ampere's Law (Steady Currents)	$\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int \mathbf{j} \cdot d\mathbf{A} = \mu_0 I_{linked}$

The integral in the middle term is over the area of a surface bounded by the closed path chosen. It is therefore the flux of **j** through that area, which is the net amount of current passing through that area. We call this the current "linked" by the path. The direction of $d\mathbf{A}$ is given by a right-hand rule: curl the fingers the way the line integral is taken; the thumb indicates the direction of $d\mathbf{A}$, which is the direction of *positive* currents.

Ampere's law as written here is valid only when the currents and the B-field are independent of time. It must be altered (by adding a term to the right side of the equation) to deal with cases where these quantities vary with time. But the magnetic Gauss's law is valid for all B-fields.

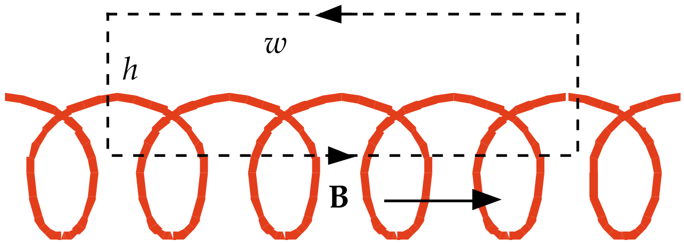
Both of these properties can be derived from the Biot-Savart law, and it can be derived from them. Therefore they are logically equivalent to the Biot-Savart law. But since carrying out the integration to use the Biot-Savart law is often difficult, the "global" properties given by Gauss's and Ampere's laws are often very useful.

Use of Ampere's Law to Calculate B

Like Gauss's law for electricity, Ampere's law can be used in cases of high symmetry to calculate B. The essential trick is to choose the path of integration so that B can be extracted from the integral on the left side.

In realistic situations the symmetry is never exact, so results obtained this way are always to some extent approximations. They are nevertheless often useful.

An important case is that of a solenoid, which is a coil of wire in the form of a helix. The situation has an obvious cylindrical symmetry. The B-field is strong inside the coil and weak outside, except near the ends. If the coil were infinite in length then all points along the axial dimension would be equivalent. If the coil's length is large compared to its diameter, the field near the middle is approximately that of an infinite coil.



Shown is a section of an infinite solenoid. We apply Ampere's law to the rectangular path shown. Along the bottom of the rectangle the line integral gives Bw. Since **B** is along the axis, it is perpendicular to the sides of the rectangle, so they give zero contribution to the integral. Outside the coil the field is (approximately) zero. The whole line integral gives Bw.

The linked current is the current in the coil (I) time the number of turns that cut through the area of the rectangle. The latter is the number of turns per unit length (n) times the width w of the rectangle. We have from Ampere's law

$$Bw = \mu_0 nwI$$

We have found a simple and useful (approximate) formula for the field near the center of a long solenoid:

$$B = \mu_0 nI$$

A better approximation to the field along the axis of a solenoid, useful even at points near the ends, is obtained by treating the coil as a set of circular loops connected in series, and using the formula given above for the field on the axis of each circular loop.

Summary of Static Fields

Here are the equations describing electric and magnetic fields, in the case of static charges and steady currents, so that the fields themselves are independent of time:

Equations for Static Fields	$\oint \mathbf{E} \cdot d\mathbf{A} = 4\pi k Q_{enc}$ $\oint \mathbf{E} \cdot d\mathbf{r} = 0$ $\oint \mathbf{B} \cdot d\mathbf{A} = 0$ $\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I_{linked}$

We see that the two fields **E** and **B** are disconnected from each other, since no equation involves both fields. Electrostatics and magnetostatics are independent parts of physics.

If the fields vary with time, we will find that there are new terms on the right sides of the second and fourth of these equations, making the E and B fields depend on each other. The resulting equations (Maxwell's equations) describe the **electromagnetic field**, of which **E** and **B** are the constituent parts.

The field equations are supplemented by the law giving the electromagnetic force on a point charge.

Lorentz Force on a Charged Particle	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

This formula remains valid even if the fields vary with time.