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AC Circuits with Complex Numbers

Addendum to Chapter 28

Introduction to Phasors

This note is not intended to replace Chapter 28 in your text, but only to introduce a different method for solving AC circuit problems. The complex variable description presented here greatly simplifies most of the derivations and problem solutions in this chapter and is also useful for the chapters on diffraction and interference of light. The methods apply to any oscillating system.

The great physicist and teacher Richard Feynman has called Euler's equation "the most important equation in mathematics." If we use the symbol j to represent $\sqrt{-1}$, this equation can be written as

$$e^{j\theta} \equiv \cos\theta + j\sin\theta \qquad 1$$

The symbol j is used here to avoid confusion with the current i. The correctness of this identity can be demonstrated by using infinite series expansions for e^x, $\sin x$, and $\cos x$.

The left side of this equation is strictly an algebraic expression, one that has very similar differential and integral forms, while the right side can be given a geometrical interpretation as seen in Fig. 1. Note that in this figure the "vector" has length one to match the factor multiplying the $e^{j\theta}$ expression on the left. The "vector" drawn is in the complex plane and is known as a phasor. Phasors can be added and subtracted like vectors, but do not share any other vector properties. For example, neither a dot or cross product is defined.

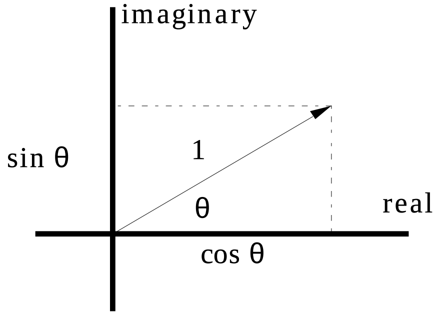


Figure 1

Notice that by using Euler's equation we can produce a number of unusual identities: $e^{j0} = 1$, $e^{j\pi} = -1$, $e^{j\pi/2} = j$, $e^{j3\pi/2} = -j$, being some of the easier ones. The complex formulation is also very useful for proving trigonometric identities for the sum and difference of two angles. Try using

Euler's equation to expand both sides of

$$e^{j\theta}e^{j\varphi} = e^{j(\theta+\varphi)} \ .$$

More general examples of a phasor would be $Ae^{j\theta}$, $Ae^{j\omega t}$, $Ae^{j(\omega t+\delta)}$ or even $A(t)e^{j(\omega t+\delta)}$ and all of these could be expanded in terms of sinusoides using Euler's equation. Except for the first, all of these change with time. Using Euler's equation, the last of these can be represented graphically as shown in Fig. 2. In this figure it can be seen that the angle ωt increases with time causing the phasor to spin counter-clockwise around the origin, like the hands of a clock going backward. The projection on the real axis is the sinusoidal term $A(t)\cos(\omega t+\delta)$ which can represent a physical oscillation.

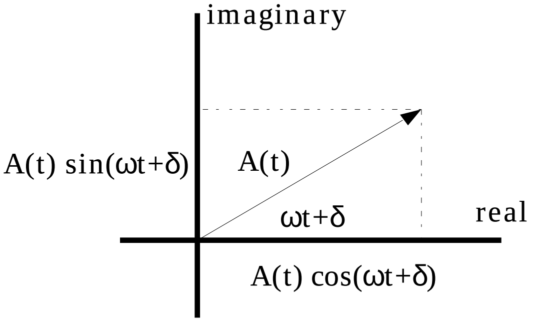


Figure 2

Simple Harmonic Oscillator

In Physics 51 your text made an analogy between uniform circular motion and the harmonic oscillator, and you will see similar representations in this AC circuits chapter and later chapters on light diffraction and interference. The geometry is the same here as in your text, but without the use of the complex plane we cannot make the connection with Euler's equation and the useful equivalency between rotating motion and a complex exponential.

The exponential expression $Ae^{j\omega t}$ on the left side of an Euler's equation is particularly useful for representing the solution of differential equations describing oscillating systems. This utility is because the derivative of $Ae^{j\omega t}$ is just $jwAe^{j\omega t}$ which has the same time variation as the original expression. Compare this with the derivative of $\cos(\omega t)$ which becomes a function with a different time dependency, namely $-\sin(\omega t)$.

Consider the simple harmonic oscillator described by $F = ma$. With $F = -kx$ this equation can be written as the differential equation

$$m\frac{d^2x}{dt^2} + kx = 0 \ .$$

The solutions to this equation all require the introduction of an "angular velocity" type variable ω defined by

$$\omega = \sqrt{\frac{k}{m}} \quad ,$$

which is known as the *natural* frequency of the oscillator.

With this definition of ω, there are several sinusoidal expressions for x(t) that are solutions to Eq. 2; one of the more general ones is

$$x(t) = C\cos(\omega t + \delta) \quad , \qquad 3$$

where *C* and δ are the constants: *C* is a real number defining the amplitude of the oscillation, and δ is the "phase factor" which determines where *x(t)* is in its cycle at $t = 0$. That this expression is a solution can be verified by forming d^2x/dt^2 and substituting it and Eq. 3 back into Eq. 2.

The phasor equivalent of Eq. 3 is

$$\boldsymbol{x}(t) = Ce^{j(\omega t + \delta)} \quad , \qquad 4$$

which can also be verified as a solution by substitution back into Eq. 2. Note that at $t = 0$, $\mathbf{x}(0) = Ce^{j\delta}$, a phasor of amplitude *C* making an angle δ with the positive real axis. This interpretation clearly shows the relationship of the phase factor to the initial conditions.

To obtain a real *x(t)* having physical meaning, we must take the real part of **x***(t)* to get

$$x(t) = \mathrm{Real}\left(Ce^{j(\omega t + \delta)}\right) = C\cos(\omega t + \delta) \quad . \qquad 5$$

Driven Harmonic Oscillator and Resonance

The differential equation for a simple LC circuit has a form similar to Eq. 2 so has solutions for v(t), i(t), and q(t) that are similar to Eq. 4. But most AC circuits also include a sinusoidal EMF adding energy to the circuit. In the mechanical system being considered here, that modification amounts to replacing the zero on the right side of Eq. 2 with a sinusoidal driving force. When a mechanical or electrical system is driven by another sinusoidal force, the system will oscillate at the frequency of the external force rather than the natural frequency. A match of the two frequencies is known as *resonance*.

The mathematics is much easier if we work with the complex representation of a driving force. Adding such a force to Eq. 2 gives

$$m\frac{d^2x}{dt^2} + kx = F_0 e^{j\omega t} \quad . \qquad 6$$

Assuming a simple solution for $x(t)$ that has $\delta = 0$

$$x(t) = Ae^{j\omega t} \quad ,$$

we can differentiate it twice to get

$$\frac{d^2x}{dt^2} = -A\omega^2 e^{j\omega t} \quad .$$

Substituting these two time dependent functions back into Eq. 6, canceling the common $e^{j\omega t}$ and solving for A gives

$$A = \frac{F_0/m}{\frac{k}{m} - \omega^2} \quad ,$$

which can also be written

$$A = \frac{F_0/m}{\omega_0^2 - \omega^2} \qquad 7$$

if we understand that ω_0 is the natural frequency of the system. As ω approaches the value ω_0 the amplitude of the oscillation becomes large: this condition is known as resonance.

Series LCR Circuit

The series LCR circuit shown in Fig. 28-15 in Tipler is described by Equation 28-46 which is essentially

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = \mathcal{E}(t) \tag{8}$$

where the EMF $\mathcal{E}(t)$ in the book is $\mathcal{E}_{max}\cos(\omega t)$. If we differentiate every term in this equation with respect to time, we get a differential equation for the current

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{d\mathcal{E}}{dt} \quad . \tag{9}$$

Now we assume that the EMF is a sinusoid just like the text, but describe it by the complex expression

$$\mathcal{E}(t) = \mathcal{E}_0 e^{j\omega t} \quad , \tag{10}$$

and understand that to get the actual EMF we must take the real part of this expression. Differentiating this expression to get the right hand side of Eq. 9 gives

$$\frac{d\mathcal{E}}{dt} = j\omega\mathcal{E}_0 e^{j\omega t} \quad . \tag{11}$$

We now assume a solution for the current that has a similar complex form

$$\mathbf{I}(t) = I_0 e^{j(\omega t - \theta)} \quad . \tag{12}$$

Here we allow the phase of the current to be different from that of the driving EMF: the minus sign of the phase factor θ is chosen for later convenience. The actual solution for the real current will be the real part of this complex representation.

The first and second derivatives of this complex current expression are

$$\frac{d\mathbf{I}}{dt} = j\omega I_0 e^{j(\omega t - \theta)}$$

and

$$\frac{d^2\mathbf{I}}{dt^2} = -\omega^2 I_0 e^{j(\omega t - \theta)} .$$

Note that because of the complex exponential notation, both derivatives have the same time dependence as $\mathbf{I}(t)$ itself. Substitution back into Eq. 9 gives

$$\mathsf{E}_0 e^{j\omega t} = \left(R + j\omega L + \frac{1}{j\omega C}\right) I_0 e^{j(\omega t - \theta)} . \qquad 13$$

The exponential part of the current expression on the right can be rewritten as a product of two exponentials

$$\mathsf{E}_0 e^{j\omega t} = \left(R + j\omega L + \frac{1}{j\omega C}\right) I_0 e^{-j\theta} e^{j\omega t} . \qquad 14$$

Note that both sides of this expression now have the same time dependence even though the original differential equation had terms in I, dI/dt and d^2I/dt^2. This cancellation would not have been possible with a real sinusoidal expression and much more complicated algebra would be necessary. Canceling the common time dependence gives

$$\mathsf{E}_0 = \left(R + j\omega L + \frac{1}{j\omega C}\right) I_0 e^{-j\theta} . \qquad 15$$

Note that since E_0 and I_0 are real, the term in parenthesis must equal some real number times $e^{+j\theta}$.

Complex Version of Ohm's Law and the Definition of Impedance

Now we extend Ohm's law V = RI to the more general complex form **V** = **ZI**, where the **Z** factor connecting a complex voltage and current is known as the impedance. In this *complex version of Ohm's law*, the complex impedance **Z** behaves mathematically much like the resistance: impedances in series and parallel behave like resistors in series and parallel.

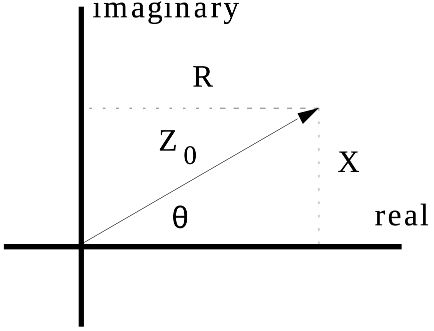


Figure 3

In Eq. 15, the complex impedance is seen to be

$$\mathbf{Z} = R + j\omega L + \frac{1}{j\omega C} \qquad 16$$

which can be rewritten as a simple complex number.

$$\mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) = R + jX \qquad 17$$

The new real quantity X is known as the reactance and is just the imaginary component of the impedance as shown in Fig. 3. Note that X contains all of the frequency dependent terms. We can also write the Eq. 17 as

$$\mathbf{Z} = Z_0 e^{j\theta}$$

which shows that as with any complex number, Euler's equation can be used to rewrite the impedance in terms of a magnitude and a phase.

Considering each term of Eq. 16 separately, we can define the impedance of a resistor R, and inductor L, and a capacitor C as

$$\mathbf{Z}_R = R$$

$$\mathbf{Z}_L = j\omega L \qquad 18$$

$$\mathbf{Z}_C = \frac{1}{j\omega C}$$

respectively. Note that impedances in series and parallel follow the rules defined for resistors. In the circuit described by Eq. 8, all three impedances are in series so they simply add to give the total impedance in the circuit. Note that even capacitive impedances require no special rule because the capacitance is in the denominator of the definition of $\mathbf{Z}_C$.

Note that the figures and equations in your text that use phasors are consistent with the definitions given here. Figures such as 28-9 and 28-16 and others simply need to be understood as having real and imaginary axes like the phasor figures in this note.

Phase Difference

The complex representation of AC voltage and current signals makes it easy to determine the relative phase between signals. The relative phase between the current through an impedance and the voltage across the impedance can be easily determined using the complex extension of Ohm's law. Since **I** and **V** can be thought of as phasors rotating counter-clockwise as time increases (so that they are both functions of time), the equation **V = Z I** can be taken to represent

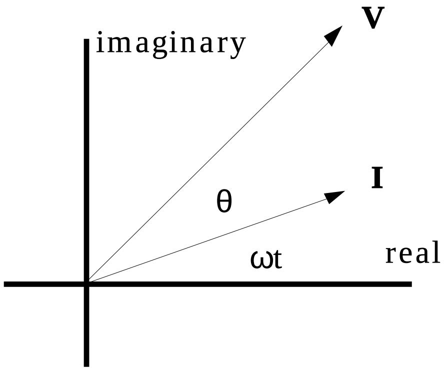


Figure 4

$$V_0 e^{j(\omega t+\theta)} = \mathbf{Z} I_0 e^{j\omega t} \quad . \qquad 19$$

Using Euler's equation, we can represent these equations as shown on Fig. 4. As shown on the figure, θ is the amount by which **V** leads **I** as both spin around the origin in a counter-clockwise direction. At some time *t*, **I** will cross the positive real axis. At this instant in time, **I** will be a real number. If we stop time at this moment (take a snapshot), the phasors are as shown in Fig. 5, and it can be seen that the phase angle of **V** with respect to the real axis is the angle between the two phasors shown in Fig. 4.

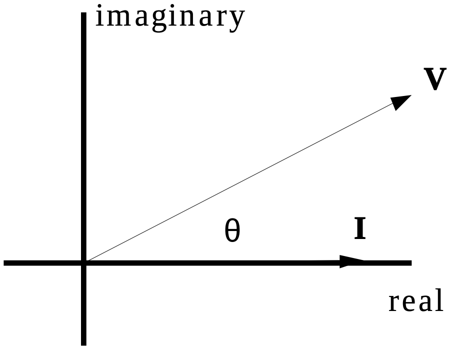


Figure 5

Problems

1. Draw a phasor on the complex plane that represents each of the following. That is, plot the complex point, then draw a phasor (vector) from the origin to the point.
   - (a) $3 + j4$
   - (b) $1/(3 + j4)$
   - (c) $4e^{j\pi/4}$
   - (d) $e^{j0.25\pi t}$ for t = 1, 2, and 3 seconds
   - (e) $3e^{j\pi/2} + 4e^{j\pi}$

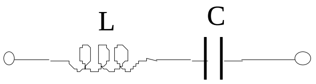
2. Use the complex form of Ohm's law to show graphically that the voltage across an inductance L leads the current through the inductor by ninety degrees.

3. Using the same method, determine the angle by which the voltage across a capacitor leads the current through the capacitor.

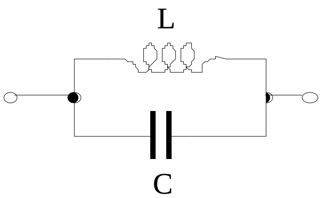
4. Let $\mathrm{E} = \mathrm{E}_0 e^{j\pi/2} e^{j\omega t}$ and $\mathbf{I} = I_0 e^{j\omega t}$.
   - (a) Plot these phasors when **I** is in a typical position in the first quadrant of the complex plane. Repeat for a time when **E** is in the first quadrant.
   - (b) By what angle does **E** lead **I**?
   - (c) Use **V** = **Z I** to obtain an expression for **Z**.
   - (d) Does this **Z** correspond to a capacitor or an inductor?

5. Write a reduced expression for the **Z** of the following combinations.

(a)



(b)



(c)