Calculating the Matrix Elements for ^{39}K

■ Determining the $\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' = 3 \mid \mu_1^1 \mid \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 2 \right\rangle$ matrix element from the measured lifetime

The lifetime of the $L = 1$ excited states in ^{39}K, τ, has been measured to be 25.8 ns. All but one of the excited states can spontaneously decay to several ground states. Because $\mid F' = 3,\ m' = 3\rangle$ decays only to $\mid F = 2,\ m = 2\rangle$, the spontaneous lifetime is related to $\mu = \left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' = 3 \mid \mu_1^1 \mid \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 2 \right\rangle$ by $\frac{1}{\tau_{sp}} = \frac{4 k^3}{3 \hbar} \mu^2$.

```
ℏ = 1.054 × 10^-27 (* erg·s *); k = 2 π / (766.7 × 10^-7) (* cm^-1 *); τ = 25.8 × 10^-9 (* s *);

μ = Sqrt[3 ℏ / (4 k^3 τ)] (* esu·cm *)

7.46121 × 10^-18
```

Generally, μ is put into Debye (D). $1\ D = 10^{-18}$ esu · cm.

```
μ = μ / 10^-18

7.46121
```

■ Calculating the $\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' \mid \mu_q^1 \mid \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m \right\rangle$ matrix elements

■ Calculating the reduced matrix element $\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 2 \right\rangle$

Using JET 92.1:

$$\langle n' (l' s)\, j'\, m' \mid \mu_q^1 \mid n (l\, s)\, j\, m \rangle = (-1)^{j' - m'} \begin{pmatrix} j' & 1 & j \\ -m', & q & m \end{pmatrix} \langle n' (l' s)\, j' \,\|\, \mu^1 \,\|\, n (l\, s)\, j \rangle$$

we can calculate the matrix elements for other transitions between these two F−levels.Knowing the left hand side of the equation for one transition allows us to calculate the reduced matrix element in the $\boldsymbol{F = I + J}$ basis. Rewriting 92.1:

$$\langle n' (J' I)\, F'\, m' \mid \mu_q^1 \mid n (J\, I)\, F\, m \rangle = (-1)^{F' - m'} \begin{pmatrix} F' & 1 & F \\ -m', & q & m \end{pmatrix} \langle n' (J' I)\, F' \,\|\, \mu^1 \,\|\, n (J\, I)\, F \rangle$$

The following function calculates the entire coefficient on the right side of the equation:

```
threeJcoeff[fp_, mp_, q_, f_, m_] := (-1)^(fp-mp) ×
    ThreeJSymbol[{fp, -mp}, {1, q}, {f, m}]
```

```
threeJcoeff[3, 3, 1, 2, 2]
```

$\frac{1}{\sqrt{7}}$

This gives $\left\langle\left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3 \,||\, \mu^1 \,||\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 2\right\rangle = \sqrt{7}\ \mu$. Using this result and 92.1 repetitively, we an determine all of the $\left\langle\left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' \,|\, \mu_q^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m\right\rangle$ matrix elements.

■ Calculating the $\left\langle\left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' \,|\, \mu_q^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m\right\rangle$ matrix elements

We will express all matrix elements in terms of μ as calculated above. The function f232 (2 for the D_2 line, 3 for $F' = 3$ and 2 for $F = 2$) performs the calculation:

```
f232[mp_, q_, m_] := threeJcoeff[3, mp, q, 2, m] √7
```

■ Calculations

```
f232[3, 1, 2]
```

1

```
f232[2, 0, 2]
```

$\frac{1}{\sqrt{3}}$

```
f232[1, -1, 2]
```

$\frac{1}{\sqrt{15}}$

```
f232[2, 1, 1]
```

$\sqrt{\frac{2}{3}}$

```
f232[1, 0, 1]
```

$2\sqrt{\frac{2}{15}}$

```
f232[0, -1, 1]
```

$\frac{1}{\sqrt{5}}$

```
f232[1, 1, 0]
```

$\sqrt{\frac{2}{5}}$

```
f232[0, 0, 0]
```

$\sqrt{\frac{3}{5}}$

■ Results in μ units

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' = 3 \mid \mu_1^1 \mid \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 2 \right\rangle = 1$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' = 2 \mid \mu_0^1 \mid \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 2 \right\rangle = \frac{1}{\sqrt{3}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' = 1 \mid \mu_{-1}^1 \mid \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 2 \right\rangle = \frac{1}{\sqrt{15}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' = 2 \mid \mu_1^1 \mid \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 1 \right\rangle = \sqrt{\frac{2}{3}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' = 1 \mid \mu_0^1 \mid \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 1 \right\rangle = 2\sqrt{\frac{2}{15}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' = 0 \mid \mu_{-1}^1 \mid \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 1 \right\rangle = \frac{1}{\sqrt{5}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' = 1 \mid \mu_1^1 \mid \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 0 \right\rangle = \sqrt{\frac{2}{5}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3,\ m' = 0 \mid \mu_0^1 \mid \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 0 \right\rangle = \sqrt{\frac{3}{5}}$

The remaining elements can be determined using the following rule:

$\langle F',\ m' \mid \mu_q^1 \mid F,\ m \rangle =$ Since

$$(-1)^{F'-m'} \begin{pmatrix} F' & 1 & F \\ -m' & q & m \end{pmatrix} \sqrt{7}\ \mu = (-1)^{-m'-F-1} \begin{pmatrix} F' & 1 & F \\ m' & -q & -m \end{pmatrix} \sqrt{7}\ \mu = (-1)^{-F'-F-2m'-1} \langle F',\ -m' \mid \mu_{-q}^1 \mid F,\ -m \rangle$$

for ^{39}K the 2×m will be even, we can write:

$\langle F',\ m' \mid \mu_q^1 \mid F,\ m \rangle = (-1)^{-F'-F-1} \langle F',\ -m' \mid \mu_{-q}^1 \mid F,\ -m \rangle$

For transitions between these two levels,the quantity $\phi = (-1)^{-F'-F-1} = (-1)^{-3-2-1} = 1$.

The matrix elements can be checked by using the sum−rule:

$\sum_{m',m} (\,|\langle F',\ m' \mid \mu_q^1 \mid F,\ m \rangle|)^2 = \frac{(|\langle F' \| \mu^1 \| F \rangle|)^2}{3} = \frac{7\,\mu}{3}$

For $q = \pm 1$:

$$1 + \frac{1}{15} + \frac{2}{3} + \frac{1}{5} + \frac{2}{5} == \frac{7}{3}$$

```
True
```

For $q = 0$:

$$2 \times \left(\frac{1}{3} + \frac{8}{15}\right) + \frac{3}{5} == \frac{7}{3}$$

```
True
```

■ Calculating the other D_2 matrix elements

■ The reduced matrix elements in the D_2 line

To calculate the matrix elements between different F−levels of the D_2 line, we can use JET 104.1:

$$\langle (J' I) F' \| \mu^1 \| (J I) F \rangle = (-1)^{J'+I+F+1} \sqrt{(2F'+1)(2F+1)} \begin{Bmatrix} J' & I & F' \\ F & 1 & J \end{Bmatrix} \langle J' \| \mu^1 \| J \rangle$$

The function sixJcoeff evaluates the coefficient on the right side of the equation:

```
sixJcoeff[jp_, i_, fp_, j_, f_] :=
  (-1)^(jp+i+f+1) Sqrt[(2 fp + 1) (2 f + 1)] SixJSymbol[{jp, i, fp}, {f, 1, j}]
```

We know $\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3 \| \mu^1 \| \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2 \right\rangle = \sqrt{7}\,\mu$, so we can calculate $\left\langle J' = \frac{3}{2} \| \mu^1 \| J = \frac{1}{2} \right\rangle$ by evaluating :

```
sixJcoeff[3 / 2, 3 / 2, 3, 1 / 2, 2]
```

$\frac{\sqrt{7}}{2}$

This gives $\left\langle J' = \frac{3}{2} \| \mu^1 \| J = \frac{1}{2} \right\rangle = 2\,\mu$. We can use this result and 104.1 again to calculate other reduced matrix elements in the D_2 line.

```
g2[fp_, f_] := sixJcoeff[3 / 2, 3 / 2, fp, 1 / 2, f] × 2
```

■ Calculations

```
g2[3, 2]
```

$\sqrt{7}$

```
g2[2, 2]
```

$-\sqrt{\frac{5}{2}}$

```
g2[1, 2]
```

$\frac{1}{\sqrt{2}}$

```
g2[2, 1]
```

$\sqrt{\frac{5}{2}}$

```
g2[1, 1]
```

$-\sqrt{\frac{5}{2}}$

```
g2[0, 1]
```

1

Results in μ units

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 3 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 2 \right\rangle = \sqrt{7}$
$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 2 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 2 \right\rangle = -\sqrt{\frac{5}{2}}$
$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 1 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 2 \right\rangle = \frac{1}{\sqrt{2}}$
$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 2 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 1 \right\rangle = \sqrt{\frac{5}{2}}$
$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 1 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 1 \right\rangle = -\sqrt{\frac{5}{2}}$
$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 0 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 1 \right\rangle = 1$
To check these results, we can use the following sum−rule:
$\sum_{F'} \left(\left| \langle (J' I) F' \,\|\, \mu^1 \,\|\, (J I) F \rangle \right| \right)^2 = \left(\left| \langle J' \,\|\, \mu^1 \,\|\, J \rangle \right| \right)^2 \, 2F + \frac{1}{2J+1}$
For $F = 2,\ J = \frac{1}{2}$, we have:

$$\left(\sqrt{7}\right)^2 + \left(-\sqrt{\frac{5}{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 == 2^2 \frac{2\,(2)\,+1}{2\,(1/2)\,+1}$$

```
True
```

For $F = 1$, $J = \frac{1}{2}$, we have:

$$\left(\sqrt{\frac{5}{2}}\right)^2 + \left(-\sqrt{\frac{5}{2}}\right)^2 + (1)^2 == 2^2 \frac{2\,(1)\,+1}{2\,(1/2)\,+1}$$

```
True
```

Calculating the $\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 2,\ m' \,|\, \mu^1_q \,|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m \right\rangle$ matrix elements

With $\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 2 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 2 \right\rangle = -\sqrt{\frac{5}{2}}$ and JET 92.1 we can calculate the matrix elements between $F' = 2$ and $F = 2$. Defining f222:

```
f222[mp_, q_, m_] := threeJcoeff[2, mp, q, 2, m] × -Sqrt[5/2]
```

Calculations

```
f222[2, 0, 2]
```

$-\frac{1}{\sqrt{3}}$

```
f222[1, -1, 2]
```

$-\frac{1}{\sqrt{6}}$

```
f222[2, 1, 1]
```

$\frac{1}{\sqrt{6}}$

```
f222[1, 0, 1]
```

$-\frac{1}{2\sqrt{3}}$

```
f222[0, -1, 1]
```

$-\frac{1}{2}$

```
f222[1, 1, 0]
```

$\frac{1}{2}$

```
f222[0, 0, 0]
```

0

Results in μ units

$$\left\langle \left(J' = \tfrac{3}{2}, I = \tfrac{3}{2}\right) F' = 2,\ m' = 2 \,|\, \mu^1_0 \,\middle|\, \left(J = \tfrac{1}{2}, I = \tfrac{3}{2}\right) F = 2,\ m = 2 \right\rangle = -\frac{1}{\sqrt{3}}$$
$$\left\langle \left(J' = \tfrac{3}{2}, I = \tfrac{3}{2}\right) F' = 2,\ m' = 1 \,|\, \mu^1_{-1} \,\middle|\, \left(J = \tfrac{1}{2}, I = \tfrac{3}{2}\right) F = 2,\ m = 2 \right\rangle = -\frac{1}{\sqrt{6}}$$
$$\left\langle \left(J' = \tfrac{3}{2}, I = \tfrac{3}{2}\right) F' = 2,\ m' = 2 \,|\, \mu^1_1 \,\middle|\, \left(J = \tfrac{1}{2}, I = \tfrac{3}{2}\right) F = 2,\ m = 1 \right\rangle = \frac{1}{\sqrt{6}}$$
$$\left\langle \left(J' = \tfrac{3}{2}, I = \tfrac{3}{2}\right) F' = 2,\ m' = 1 \,|\, \mu^1_0 \,\middle|\, \left(J = \tfrac{1}{2}, I = \tfrac{3}{2}\right) F = 2,\ m = 1 \right\rangle = -\frac{1}{2\sqrt{3}}$$
$$\left\langle \left(J' = \tfrac{3}{2}, I = \tfrac{3}{2}\right) F' = 2,\ m' = 0 \,|\, \mu^1_{-1} \,\middle|\, \left(J = \tfrac{1}{2}, I = \tfrac{3}{2}\right) F = 2,\ m = 1 \right\rangle = -\frac{1}{2}$$
$$\left\langle \left(J' = \tfrac{3}{2}, I = \tfrac{3}{2}\right) F' = 2,\ m' = 1 \,|\, \mu^1_1 \,\middle|\, \left(J = \tfrac{1}{2}, I = \tfrac{3}{2}\right) F = 2,\ m = 0 \right\rangle = \frac{1}{2}$$
$$\left\langle \left(J' = \tfrac{3}{2}, I = \tfrac{3}{2}\right) F' = 2,\ m' = 0 \,|\, \mu^1_0 \,\middle|\, \left(J = \tfrac{1}{2}, I = \tfrac{3}{2}\right) F = 2,\ m = 0 \right\rangle = 0$$

The ϕ for these two levels is $\phi = (-1)^{-2-2-1} = -1$.

Checking these results using the sum−rule $\sum_{m',m} (\,|\langle F',\ m' \,|\, \mu^1_q \,|\, F,\ m\rangle|)^2 = \frac{(|\langle F' \| \mu^1 \| F\rangle|)^2}{3}$:
For $q = \pm 1$:

$$\left(\frac{-1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 == \frac{1}{3}\left(-\sqrt{\frac{5}{2}}\right)^2$$

```
True
```

For q=0:

$$2\left(\left(\frac{-1}{\sqrt{3}}\right)^2 + \left(\frac{-1}{2\sqrt{3}}\right)^2\right) + 0 == \frac{1}{3}\left(-\sqrt{\frac{5}{2}}\right)^2$$

```
True
```

■ Calculating the $\left\langle\left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 1,\ m' \,|\, \mu^1_q \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m\right\rangle$ matrix elements

$$\texttt{f212[mp\_, q\_, m\_] := threeJcoeff[1, mp, q, 2, m]} \times \frac{1}{\sqrt{2}}$$

■ Calculations

```
f212[1, -1, 2]
```

$$\frac{1}{\sqrt{10}}$$

```
f212[1, 0, 1]
```

$$-\frac{1}{2\sqrt{5}}$$

```
f212[0, -1, 1]
```

$$\frac{1}{2\sqrt{5}}$$

```
f212[1, 1, 0]
```

$$\frac{1}{2\sqrt{15}}$$

```
f212[0, 0, 0]
```

$$-\frac{1}{\sqrt{15}}$$

■ Results in μ units

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 1,\ m' = 1 \right| \mu^1_{-1} \left| \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 2 \right\rangle = \frac{1}{\sqrt{10}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 1,\ m' = 1 \right| \mu^1_{0} \left| \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 1 \right\rangle = -\frac{1}{2\sqrt{5}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 1,\ m' = 0 \right| \mu^1_{-1} \left| \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 1 \right\rangle = \frac{1}{2\sqrt{5}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 1,\ m' = 1 \right| \mu^1_{1} \left| \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 0 \right\rangle = \frac{1}{2\sqrt{15}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 1,\ m' = 0 \right| \mu^1_{0} \left| \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 0 \right\rangle = -\frac{1}{\sqrt{15}}$

$\phi = (-1)^{-1-2-1} = 1$

$$\left(\frac{1}{\sqrt{10}}\right)^2 + \left(\frac{1}{2\sqrt{5}}\right)^2 + \left(\frac{1}{2\sqrt{15}}\right)^2 == \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^2$$

```
True
```

$$2\left(-\frac{1}{2\sqrt{5}}\right)^2 + \left(\frac{-1}{\sqrt{15}}\right)^2 == \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^2$$

```
True
```

■ Calculating the $\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 2,\ m' \right| \mu^1_q \left| \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m \right\rangle$ matrix elements

```
f221[mp_, q_, m_] := threeJcoeff[2, mp, q, 1, m] × Sqrt[5/2]
```

■ Calculations

```
f221[2, 1, 1]
```

$\frac{1}{\sqrt{2}}$

```
f221[1, 0, 1]
```

$\frac{1}{2}$

```
f221[0, -1, 1]
```

$\frac{1}{2\sqrt{3}}$

```
f221[1, 1, 0]
```

$\frac{1}{2}$

```
f221[0, 0, 0]
```

$\frac{1}{\sqrt{3}}$

Results in terms of μ

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 2 \,|\, \mu_1^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 1 \right\rangle = \frac{1}{\sqrt{2}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 1 \,|\, \mu_0^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 1 \right\rangle = \frac{1}{2}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 0 \,|\, \mu_{-1}^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 1 \right\rangle = \frac{1}{2\sqrt{3}}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 1 \,|\, \mu_1^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 0 \right\rangle = \frac{1}{2}$

$\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 0 \,|\, \mu_0^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 0 \right\rangle = \frac{1}{\sqrt{3}}$

$\phi = (-1)^{-2-1-1} = 1$

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{3}}\right)^2 + \left(\frac{1}{2}\right)^2 == \frac{1}{3}\left(\sqrt{\frac{5}{2}}\right)^2$$

```
True
```

$$2\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 == \frac{1}{3}\left(\sqrt{\frac{5}{2}}\right)^2$$

```
True
```

Calculating the $\left\langle \left(J' = \frac{3}{2}, I = \frac{3}{2}\right) F' = 1,\ m' \,|\, \mu_q^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m \right\rangle$ matrix elements

$$\texttt{f211[mp\_, q\_, m\_] := threeJcoeff[1, mp, q, 1, m]} \times -\sqrt{\frac{5}{2}}$$

Calculations

```
f211[1, 0, 1]
```

$-\frac{\sqrt{\frac{5}{3}}}{2}$

```
f211[0, -1, 1]
```

$-\frac{\sqrt{\frac{5}{3}}}{2}$

```
f211[1, 1, 0]
```

$\frac{\sqrt{\frac{5}{3}}}{2}$

```
f211[0, 0, 0]
```

```
0
```

Results in μ units

$\langle (J' = \frac{3}{2}, I = \frac{3}{2}) F' = 1,\ m' = 1 | \mu^1_0 | (J = \frac{1}{2}, I = \frac{3}{2}) F = 1,\ m = 1 \rangle = -\frac{\sqrt{\frac{5}{3}}}{2}$

$\langle (J' = \frac{3}{2}, I = \frac{3}{2}) F' = 1,\ m' = 0 | \mu^1_{-1} | (J = \frac{1}{2}, I = \frac{3}{2}) F = 1,\ m = 1 \rangle = -\frac{\sqrt{\frac{5}{3}}}{2}$

$\langle (J' = \frac{3}{2}, I = \frac{3}{2}) F' = 1,\ m' = 1 | \mu^1_1 | (J = \frac{1}{2}, I = \frac{3}{2}) F = 1,\ m = 0 \rangle = \frac{\sqrt{\frac{5}{3}}}{2}$

$\langle (J' = \frac{3}{2}, I = \frac{3}{2}) F' = 1,\ m' = 0 | \mu^1_0 | (J = \frac{1}{2}, I = \frac{3}{2}) F = 1,\ m = 0 \rangle = 0$

$\phi = (-1)^{-1-1-1} = -1$

$$\left(\frac{-1}{2}\sqrt{\frac{5}{3}}\right)^2 + \left(\frac{1}{2}\sqrt{\frac{5}{3}}\right)^2 == \frac{1}{3}\left(-\sqrt{\frac{5}{2}}\right)^2$$

```
True
```

$$2\left(\frac{-1}{2}\sqrt{\frac{5}{3}}\right)^2 + 0 == \frac{1}{3}\left(-\sqrt{\frac{5}{2}}\right)^2$$

```
True
```

Calculating the $\langle (J' = \frac{3}{2}, I = \frac{3}{2}) F' = 0,\ m' | \mu^1_q | (J = \frac{1}{2}, I = \frac{3}{2}) F = 1,\ m \rangle$ matrix elements

```
f201[mp_, q_, m_] := threeJcoeff[0, mp, q, 1, m] × 1
```

Calculations

```
f201[0, -1, 1]
```

$\frac{1}{\sqrt{3}}$

```
f201[0, 0, 0]
```

$-\frac{1}{\sqrt{3}}$

Results in μ units

$\langle (J' = \frac{3}{2}, I = \frac{3}{2}) F' = 0,\ m' = 0 | \mu^1_{-1} | (J = \frac{1}{2}, I = \frac{3}{2}) F = 1,\ m = 1 \rangle = \frac{1}{\sqrt{3}}$

$\langle (J' = \frac{3}{2}, I = \frac{3}{2}) F' = 0,\ m' = 0 | \mu^1_0 | (J = \frac{1}{2}, I = \frac{3}{2}) F = 1,\ m = 0 \rangle = -\frac{1}{\sqrt{3}}$

$\phi = (-1)^{-1-1-1} = -1$

$$\left(\frac{1}{\sqrt{3}}\right)^2 == \frac{1}{3}(1)^2$$

```
True
```

```
(-1/√3)^2 == 1/3 (1)^2
```

True

■ Calculating the D_1 matrix elements

■ Determining the $\langle(l' = 1, s = \frac{1}{2})\, J' = \frac{1}{2} \,||\, \mu^1 \,||\, (l = 0, s = \frac{1}{2})\, J = \frac{1}{2}\rangle$ matrix element

To get $\langle(l' = 1, s = \frac{1}{2})\, J' = \frac{1}{2} \,||\, \mu^1 \,||\, (l = 0, s = \frac{1}{2})\, J = \frac{1}{2}\rangle$ and the reduced matrix elements in the D_1 line, we must reduce the $\langle(l' = 1, s = \frac{1}{2})\, J' = \frac{3}{2} \,||\, \mu^1 \,||\, (l = 0, s = \frac{1}{2})\, J = \frac{1}{2}\rangle$ to the l basis usingJET 104.1 again:
$\langle(l'\, s)\, J' \,||\, \mu^1 \,||\, (l\, s)\, J\rangle = (-1)^{l'+s+J+1} \sqrt{(2\,J' + 1)\,(2\,J + 1)} \begin{Bmatrix} l' & s & J' \\ J & 1 & l \end{Bmatrix} \langle l' = 1 \,||\, \mu^1 \,||\, l = 0\rangle$

```
sixJcoeff[1, 1/2, 3/2, 0, 1/2]
```

$\frac{2}{\sqrt{3}}$

With $\langle(l' = 1, s = \frac{1}{2})\, J' = \frac{3}{2} \,||\, \mu^1 \,||\, (l = 0,\; s = \frac{1}{2})\, J = \frac{1}{2}\rangle = 2\,\mu$:

$\langle l' = 1 \,||\, \mu^1 \,||\, l = 0\rangle = \sqrt{3}\;\mu$.

Then, in μ units, $\langle(l' = 1, s = \frac{1}{2})\, J' = \frac{1}{2} \,||\, \mu^1 \,||\, (l = 0,\; s = \frac{1}{2})\, J = \frac{1}{2}\rangle$ is:

```
sixJcoeff[1, 1/2, 1/2, 0, 1/2] × √3
```

$-\sqrt{2}$

$\langle(l' = 1, s = \frac{1}{2})\, J' = \frac{1}{2} \,||\, \mu^1 \,||\, (l = 0,\; s = \frac{1}{2})\, J = \frac{1}{2}\rangle = -\sqrt{2}\;\mu$

■ The reduced matrix elements in the D_1 line

With $\langle(l' = 1, s = \frac{1}{2})\, J' = \frac{1}{2} \,||\, \mu^1 \,||\, (l = 0,\; s = \frac{1}{2})\, J = \frac{1}{2}\rangle = -\sqrt{2}\,\mu$, the D_1 reduced matrix elements are given by $\langle(J'\, I)\, F' \,||\, \mu^1 \,||\, (J\, I)\, F\rangle = (-1)^{J'+I+F+1} \sqrt{(2\,F' + 1)\,(2\,F + 1)} \begin{Bmatrix} J' & I & F' \\ F & 1 & J \end{Bmatrix} \langle J' \,||\, \mu^1 \,||\, J\rangle$ using the function g1:

```
g1[fp_, f_] := sixJcoeff[1/2, 3/2, fp, 1/2, f] × (-√2)
```

■ Calculations

```
g1[2, 2]
```

$-\sqrt{\frac{5}{2}}$

```
g1[1, 2]
```

$\sqrt{\frac{5}{2}}$

```
g1[2, 1]
```

$-\sqrt{\frac{5}{2}}$

```
g1[1, 1]
```

$\frac{1}{\sqrt{2}}$

■ Results

$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 2 \right\rangle = -\sqrt{\frac{5}{2}}$
$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 1 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 2 \right\rangle = \sqrt{\frac{5}{2}}$
$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 1 \right\rangle = -\sqrt{\frac{5}{2}}$
$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 1 \,\|\, \mu^1 \,\|\, \left(J = \frac{1}{2},\ I = \frac{3}{2}\right) F = 1 \right\rangle = \frac{1}{\sqrt{2}}$
Checking with the sum−rule:
$\sum_{F'} (\,|\langle (J'\ I)\, F' \,\|\, \mu^1 \,\|\, (J\ I)\, F\rangle|)^2 = (\,|\langle J' \,\|\, \mu^1 \,\|\, J\rangle|)^2\ \frac{2F+1}{2J+1}$
For $F = 2,\ J = \frac{1}{2}$, we have:

$$\left(-\sqrt{\frac{5}{2}}\right)^2 + \left(\sqrt{\frac{5}{2}}\right)^2 == \left(-\sqrt{2}\right)^2\ \frac{2\,(2) + 1}{2\left(\frac{1}{2}\right) + 1}$$

```
True
```

For $F = 1,\ J = \frac{1}{2}$, we have:

$$\left(-\sqrt{\frac{5}{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 == \left(-\sqrt{2}\right)^2\ \frac{2\,(1) + 1}{2\left(\frac{1}{2}\right) + 1}$$

```
True
```

■ Calculating the $\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' \,|\, \mu_q^1 \,|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m \right\rangle$ matrix elements

$$\texttt{f122[mp\_, q\_, m\_] := threeJcoeff[2, mp, q, 2, m]} \times \left(-\sqrt{\frac{5}{2}}\right)$$

Calculations

```
f122[2, 0, 2]
```

$-\frac{1}{\sqrt{3}}$

```
f122[1, -1, 2]
```

$-\frac{1}{\sqrt{6}}$

```
f122[2, 1, 1]
```

$\frac{1}{\sqrt{6}}$

```
f122[1, 0, 1]
```

$-\frac{1}{2\sqrt{3}}$

```
f122[0, -1, 1]
```

$-\frac{1}{2}$

```
f122[1, 1, 0]
```

$\frac{1}{2}$

```
f122[0, 0, 0]
```

0

Results

$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 2 \,\middle|\, \mu_0^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 2 \right\rangle = -\frac{1}{\sqrt{3}}$

$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 1 \,\middle|\, \mu_{-1}^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 2 \right\rangle = \frac{-1}{\sqrt{6}}$

$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 2 \,\middle|\, \mu_1^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 1 \right\rangle = \frac{1}{\sqrt{6}}$

$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 1 \,\middle|\, \mu_0^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 1 \right\rangle = -\frac{1}{2\sqrt{3}}$

$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 0 \,\middle|\, \mu_{-1}^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 1 \right\rangle = \frac{-1}{2}$

$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 1 \,\middle|\, \mu_1^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 0 \right\rangle = \frac{1}{2}$

$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 0 \,\middle|\, \mu_0^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 2,\ m = 0 \right\rangle = 0$

The ϕ for these two levels is $\phi = (-1)^{-2-2-1} = -1$.

$$\left(\frac{-1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 == \frac{1}{3}\left(-\sqrt{\frac{5}{2}}\right)^2$$

True

$$2\left(\left(-\frac{1}{\sqrt{3}}\right)^2+\left(-\frac{1}{2\sqrt{3}}\right)^2\right)+(0)^2==\frac{1}{3}\left(-\sqrt{\frac{5}{2}}\right)^2$$

True

■ Calculating the $\left\langle\left(J'=\frac{1}{2},I=\frac{3}{2}\right)F'=1,\ m'\,|\,\mu^1_q\,\middle|\left(J=\frac{1}{2},I=\frac{3}{2}\right)F=2,\ m\right\rangle$ matrix elements

```
f112[mp_, q_, m_] := threeJcoeff[1, mp, q, 2, m] × Sqrt[5/2]
```

■ Calculations

```
f112[1, -1, 2]
```

$\frac{1}{\sqrt{2}}$

```
f112[1, 0, 1]
```

$-\frac{1}{2}$

```
f112[0, -1, 1]
```

$\frac{1}{2}$

```
f112[1, 1, 0]
```

$\frac{1}{2\sqrt{3}}$

```
f112[0, 0, 0]
```

$-\frac{1}{\sqrt{3}}$

■ Results

$\left\langle\left(J'=\frac{1}{2},I=\frac{3}{2}\right)F'=1,\ m'=1\,|\,\mu^1_{-1}\,\middle|\left(J=\frac{1}{2},I=\frac{3}{2}\right)F=2,\ m=2\right\rangle=\frac{1}{\sqrt{2}}$
$\left\langle\left(J'=\frac{1}{2},I=\frac{3}{2}\right)F'=1,\ m'=1\,|\,\mu^1_{0}\,\middle|\left(J=\frac{1}{2},I=\frac{3}{2}\right)F=2,\ m=1\right\rangle=-\frac{1}{2}$
$\left\langle\left(J'=\frac{1}{2},I=\frac{3}{2}\right)F'=1,\ m'=0\,|\,\mu^1_{-1}\,\middle|\left(J=\frac{1}{2},I=\frac{3}{2}\right)F=2,\ m=1\right\rangle=\frac{1}{2}$
$\left\langle\left(J'=\frac{1}{2},I=\frac{3}{2}\right)F'=1,\ m'=1\,|\,\mu^1_{1}\,\middle|\left(J=\frac{1}{2},I=\frac{3}{2}\right)F=2,\ m=0\right\rangle=\frac{1}{2\sqrt{3}}$
$\left\langle\left(J'=\frac{1}{2},I=\frac{3}{2}\right)F'=1,\ m'=0\,|\,\mu^1_{0}\,\middle|\left(J=\frac{1}{2},I=\frac{3}{2}\right)F=2,\ m=0\right\rangle=-\frac{1}{\sqrt{3}}$
$\phi=(-1)^{-1-2-1}=1$

$$\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2\sqrt{3}}\right)^2==\frac{1}{3}\left(\sqrt{\frac{5}{2}}\right)^2$$

True

```
2 (-1/2)^2 + (-1/Sqrt[3])^2 == 1/3 (Sqrt[5/2])^2
```

$$2\left(\frac{-1}{2}\right)^2+\left(\frac{-1}{\sqrt{3}}\right)^2==\frac{1}{3}\left(\sqrt{\frac{5}{2}}\right)^2$$

True

■ Calculating the $\left\langle\left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' \mid \mu_q^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m\right\rangle$ matrix elements

■ Calculations

$$\text{f121[mp\_, q\_, m\_] := threeJcoeff[2, mp, q, 1, m]} \times \left(-\sqrt{\frac{5}{2}}\right)$$

f121[2, 1, 1]

$-\frac{1}{\sqrt{2}}$

f121[1, 0, 1]

$-\frac{1}{2}$

f121[0, -1, 1]

$-\frac{1}{2\sqrt{3}}$

f121[1, 1, 0]

$-\frac{1}{2}$

f121[0, 0, 0]

$-\frac{1}{\sqrt{3}}$

■ Results

$\left\langle\left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 2 \mid \mu_1^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 1\right\rangle = -\frac{1}{\sqrt{2}}$

$\left\langle\left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 1 \mid \mu_0^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 1\right\rangle = -\frac{1}{2}$

$\left\langle\left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 0 \mid \mu_{-1}^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 1\right\rangle = -\frac{1}{2\sqrt{3}}$

$\left\langle\left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 1 \mid \mu_1^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 0\right\rangle = -\frac{1}{2}$

$\left\langle\left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 2,\ m' = 0 \mid \mu_0^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 0\right\rangle = -\frac{1}{\sqrt{3}}$

$\phi = (-1)^{-2-1-1} = 1$

$$\left(\frac{-1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{2\sqrt{3}}\right)^2 + \left(\frac{-1}{2}\right)^2 == \frac{1}{3}\left(-\sqrt{\frac{5}{2}}\right)^2$$

True

$$2\left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{\sqrt{3}}\right)^2 == \frac{1}{3}\left(-\sqrt{\frac{5}{2}}\right)^2$$

True

■ Calculating the $\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 1,\ m' \,|\, \mu_q^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m\right\rangle$ matrix elements

```
f111[mp_, q_, m_] := threeJcoeff[1, mp, q, 1, m] × 1/√2
```

■ Calculations

```
f111[1, 0, 1]
```

$\frac{1}{2\sqrt{3}}$

```
f111[0, -1, 1]
```

$\frac{1}{2\sqrt{3}}$

```
f111[1, 1, 0]
```

$-\frac{1}{2\sqrt{3}}$

```
f111[0, 0, 0]
```

0

■ Results

$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 1,\ m' = 1 \,|\, \mu_0^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 1\right\rangle = \frac{1}{2\sqrt{3}}$
$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 1,\ m' = 0 \,|\, \mu_{-1}^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 1\right\rangle = \frac{1}{2\sqrt{3}}$
$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 1,\ m' = 1 \,|\, \mu_1^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 0\right\rangle = -\frac{1}{2\sqrt{3}}$
$\left\langle \left(J' = \frac{1}{2}, I = \frac{3}{2}\right) F' = 1,\ m' = 0 \,|\, \mu_0^1 \,\middle|\, \left(J = \frac{1}{2}, I = \frac{3}{2}\right) F = 1,\ m = 0\right\rangle = 0$
$\phi = (-1)^{-1-1-1} = -1$

$$\left(\frac{1}{2\sqrt{3}}\right)^2 + \left(\frac{-1}{2\sqrt{3}}\right)^2 == \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^2$$

True

$$2\left(\frac{1}{2\sqrt{3}}\right)^2 + (0)^2 == \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^2$$

```
True
```