

Hamiltonian formulation of quantum error correction and correlated noise: Effects of syndrome extraction in the long-time limit

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(Received 16 October 2007; revised manuscript received 1 April 2008; published 9 July 2008)

We analyze the long-time behavior of a quantum computer running a quantum error correction (QEC) code in the presence of a correlated environment. Starting from a Hamiltonian formulation of realistic noise models, and assuming that QEC is indeed possible, we find formal expressions for the probability of a given syndrome history and the associated residual decoherence encoded in the reduced density matrix. Systems with nonzero gate times (“long gates”) are included in our analysis by using an upper bound on the noise. In order to introduce the local error probability for a qubit, we assume that propagation of signals through the environment is slower than the QEC period (hypercube assumption). This allows an explicit calculation in the case of a generalized spin-boson model and a quantum frustration model. The key result is a dimensional criterion: If the correlations decay sufficiently fast, the system evolves toward a stochastic error model for which the threshold theorem of fault-tolerant quantum computation has been proven. On the other hand, if the correlations decay slowly, the traditional proof of this threshold theorem does not hold. This dimensional criterion bears many similarities to criteria that occur in the theory of quantum phase transitions.

DOI: [10.1103/PhysRevA.78.012314](https://doi.org/10.1103/PhysRevA.78.012314)

PACS number(s): 03.67.Lx, 03.67.Pp, 03.65.Yz, 73.21.-b

I. INTRODUCTION

Quantum computation provides a fundamentally new way to process data; as a theory, it is complete and remarkably rich [1]. However, any real quantum computer is subject to an implacable physical reality: Components of a computer will always be faulty due to environmental noise. Hence, the builder of a quantum computer faces the conundrum of having to isolate the device from its surroundings and, simultaneously, of needing to act on it and read its output [2]. Many strategies have been devised to address this problem [1,3–7], the most general being quantum error correction [1,8–11].

Quantum error correction (QEC) should be understood as a perturbative approach [12], where one can estimate the probability of having an “error” in the wave function of the quantum computer after a certain time. It is naturally formulated as a perturbation expansion in powers of the coupling between the computer and the environment [12]. QEC cannot, in general, perfectly correct the quantum evolution, and the interference of the amplitudes for the various processes that occur implies that quantum information is always lost to the environment [12]. However, as we discuss below, QEC can very effectively slow down this loss. In fact, a central theoretical result is the “threshold theorem:” It states that if the error probability is smaller than a critical value, quantum computation can be sustained indefinitely [13–21]. The word “indefinitely” deserves some clarification: For the problems that we discuss, it means that given a calculation and a desired precision, it is always possible to construct a quantum circuit that will provide the correct result with high enough probability.

QEC has been largely developed using phenomenological “error models.” Rarely is a connection to a microscopic quantum dynamical system found in the literature (see, however, Refs. [22–25]). In contrast, here we pursue exactly such a connection: We discuss the formal steps needed to link the

theory of error correction with microscopic Hamiltonian models. Furthermore, because of the perturbative nature of the method, it is possible to draw a close parallel between the “threshold theorem” and the theory of quantum phase transitions. We find that if a certain inequality holds, an error threshold always exists. When the inequality is not satisfied, either a new version of the threshold criterion is required or fault tolerant quantum computation is not possible at all. For the moment, we are not able to distinguish between these two possibilities.

Our analysis is based on the following assumptions. First and foremost, we assume that it is possible to perform the building blocks of quantum error correction, namely, preparation of states, quantum gates, and measurements. Second, we consider that the environment is described by a free field theory in which thermal fluctuations can be effectively suppressed. Finally, the main simplifying assumption of our discussion is that the qubits are sufficiently separated in space for an entire error correction procedure to be performed before correlations between nearby qubits develop. The probability of an error in an individual qubit within a QEC cycle is, therefore, independent of all other qubits. This does not imply that there are no spatial correlations; rather, they develop on longer time scales, while the error correction procedure is done faster than a certain characteristic time. We emphasize that this hypothesis is not a limitation of the general theoretical framework that we describe, but simply a way to connect to the traditional proofs of the “threshold theorem” in terms of stochastic error models.

The paper is organized as follows. Because of the interdisciplinary nature of the subject, this introduction continues with a discussion of two points. First, the difficulties in taking into account correlations in the environment are explained in Sec. I A from a perturbative point of view. Then, in Sec. I B, we discuss the QEC method from a physics viewpoint and present some results for the standard stochas-

tic error model. We start the main part of the paper by developing the relation between error models and quantum codes (Sec. II). The key issue of QEC in a correlated environment is treated in Sec. III. Our main results delineating when the perturbative treatment is valid appear in Sec. IV. At the end of this section, we provide a brief comparison between our results and those of Ref. [26]. Section V discusses parallels between the threshold theorem of QEC and the theory of quantum phase transitions. Finally, in Sec. VI we summarize our results and comment on some open problems.

A. Problem of correlated environments

In order to set the stage for the analysis in the presence of QEC, we first look at the problem of errors created by a correlated environment in an unprotected system. In the Schrödinger equation governing the time evolution of a quantum system, the Hamiltonian H can usually be separated into a single-particle term H_0 and a many-particle interaction part V . A formal solution of this equation is given by the Dyson series in the interaction picture. Solution by iteration shows that the time-evolution operator is

$$U(t,0) = T_t e^{-i\hbar \int_0^t V(t') dt'}, \quad (1)$$

with T_t denoting the time-ordering operator and $V(t) = e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar}$. If V represents the interaction between the quantum computer and its surroundings, each insertion of V in Eq. (1) corresponds to an “error” in the computer evolution. Hence, Eq. (1) provides the natural framework to study the effects of the environment on the state of the quantum computer.

It is always possible to give an upper bound to the “error probability” [27]. The reason is that Dyson’s series is absolutely convergent for finite times and bounded operators (see Appendix A). In short, the bounding is done by defining the “sup” operator norm and the evolution operator with at least one “error” (one insertion of V),

$$\mathcal{E}(t) = U(t,0) - 1 = -\frac{i}{\hbar} \int_0^t dt' V(t') U(t',0). \quad (2)$$

The norm of \mathcal{E} is related to the probability of having errors in the computer. The calculation is simple and yields

$$\|\mathcal{E}(t)\| \leq \frac{1}{\hbar} \int_0^t dt' \|V(t')\| \leq \frac{\Lambda t}{\hbar}, \quad (3)$$

where we used the triangular inequality, the unitarity of U , and defined Λ as the largest eigenvalue of V (with corresponding eigenvector Ψ_Λ). One can understand this bound as simply a restatement of $|\sin x| \leq |x|$, as follows:

$$\mathcal{E}^\dagger(t)\mathcal{E}(t) = [2 - U^\dagger(t) - U(t)] = 2 \left(1 - T_t \cos \frac{1}{\hbar} \int_0^t dt' V(t') \right), \quad (4)$$

so that

$$\sqrt{\langle \Psi_\Lambda | \mathcal{E}^\dagger(t)\mathcal{E}(t) | \Psi_\Lambda \rangle} = \sqrt{2 \left(1 - \cos \frac{\Lambda t}{\hbar} \right)} = \left| \sin \frac{\Lambda t}{2\hbar} \right| \leq \left| \frac{\Lambda t}{2\hbar} \right|. \quad (5)$$

The norm $\|\mathcal{E}\|$ has been very useful in problems involving non-Markovian noise [26–30]. However, in QEC, an analysis based on the bound equation (3) only makes sense when $\|\mathcal{E}\| \ll 1$, while we are concerned with the long-time limit, $|\Lambda t| \gg 1$, for which this bound on the norm of the error diverges. In this case, Dyson’s series is only asymptotically convergent and the “sup” norm is of no practical use. Hence, it is important to express the error probability differently.

We must go back full circle and reexamine the Dyson series for the time evolution of a particular state, instead of the worst case scenario explored by the “sup” norm approach. Henceforth, we will be mainly interested in an interaction Hamiltonian with the general form

$$V(t) = \lambda \int_0^L d\mathbf{x} f(\mathbf{x}, t), \quad (6)$$

where $\lambda \ll 1$ is a coupling constant, L is the size of the system, and f is some function of the degrees of freedom of a free theory whose Hamiltonian is H_0 . Because we are interested in correlated non-Markovian noise, we assume that the free fields are such that the asymptotic expression for the two-point correlation function is a power law,

$$\langle \Psi | f(\mathbf{x}_1, t_1) f(\mathbf{x}_2, t_2) | \Psi \rangle \sim \mathcal{F} \left(\frac{1}{(\Delta x)^{2\delta}}, \frac{1}{(\Delta t)^{2\delta/z}} \right), \quad (7)$$

where $\Delta x = |\mathbf{x}_1 - \mathbf{x}_2|$ and $\Delta t = |t_1 - t_2|$ [31]. Here, δ is the scaling dimension of f , z is the so-called dynamical exponent, and $|\Psi\rangle$ is a fixed eigenstate of H_0 (which we will usually take to be the ground state of the environment).

The motivation for developing a perturbative expansion of the evolution operator (the Dyson series in the interaction picture) is the hope that a few terms in the series or a summable family of them will capture most of the physics. It is then assumed that small coupling can guarantee fast convergence. However, since $\|\mathcal{E}\|$ is not necessarily small, the number of terms that contribute substantially to the series can grow faster than the smallness of consecutive terms. In order to see that, let us calculate the probability of an evolution with errors using Eq. (4),

$$\frac{\langle \Psi | \mathcal{E}^\dagger(t)\mathcal{E}(t) | \Psi \rangle}{2} = 1 - \langle \Psi | T_t \cos \left(\frac{1}{\hbar} \int_0^t dt' V(t') \right) | \Psi \rangle. \quad (8)$$

Since we are assuming a noninteracting free Hamiltonian, we can use Wick’s theorem. It is then straightforward to show that there is at least one term at each order m in the series that contributes “extensively” as $\sim \lambda^{2m} (L t)^{2m(D+z-\delta)}$. A simple example is given by the series of “bubble” diagrams, where the m th-order term is given by the contractions

$$\int_0^t dt_1 \cdots \int_0^{t_{m-1}} dt_m \langle V(t_1)V(t_2) \rangle \cdots \langle V(t_{m-1})V(t_m) \rangle. \quad (9)$$

Disregarding numerical prefactors unimportant for our discussion, we sum the series as a geometric progression to obtain

$$\frac{\langle \Psi | F^\dagger(t) F(t) | \Psi \rangle}{2} \sim \frac{\lambda^2 (Lt)^{2(D+z-\delta)}}{1 + \lambda^2 (Lt)^{2(D+z-\delta)}}. \quad (10)$$

Therefore, for $D+z-\delta > 0$ there is no guarantee that the perturbation series converges. Conversely, if $D+z-\delta < 0$, higher-order terms in the series should be increasingly less important. Thus, for $D+z-\delta > 0$ the probability of an evolution with “errors” tends to one, whereas for $D+z-\delta < 0$ it will depend only on the “nonextensive” terms in the series. The same analysis can be immediately transported to the study of the fidelity $|\langle \Psi | U(t) | \Psi \rangle|$, where we see that for a relevant perturbation, $D+z-\delta > 0$, the overlap between the initial state and the evolving wave function tends to zero (an orthogonality catastrophe). This sort of “infrared” problem provides a contact point with the theory of quantum phase transitions, where the same kind of considerations also appear when calculating the partition function using the imaginary time formalism (see Appendix B).

In the body of this paper, our main goal is to transfer these ideas of relevance and irrelevance of a perturbation to the evolution of a quantum computer protected by QEC.

B. Quantum error correction

Quantum error correction is arguably the most versatile method to protect quantum information from decoherence [32]. It is a clever use of two features of quantum mechanics: Entanglement and (in its traditional form) wave-packet reduction due to measurement. Thus, before we start our discussion of QEC, it is important to carefully define what we mean by entanglement and decoherence.

An entangled state of two quantum systems is a state that cannot be described as a direct tensor product of states of individual systems or probabilistic mixtures of tensor-product states. As an example, consider two physical qubits (hereafter referred to by the subscripts 1 and 2). Each qubit has a Hilbert space isomorphic to a complex projective plane of dimension 1, $\mathbb{C}P^1$ (see Appendix C for details). However, the combined Hilbert space is not isomorphic to $\mathbb{C}P^1_{(1)} \times \mathbb{C}P^1_{(2)}$, but to the much larger $\mathbb{C}P^3$. All states in $\mathbb{C}P^3$ outside $\mathbb{C}P^1_{(1)} \times \mathbb{C}P^1_{(2)}$ are said to be entangled. An important subtlety is the implicit notion of a preferred “basis.” Although we can choose from an infinite number of $\mathbb{C}P^1 \times \mathbb{C}P^1$ subspaces inside the same $\mathbb{C}P^3$, nature gives us a natural choice, namely, $\mathbb{C}P^1_{(1)} \times \mathbb{C}P^1_{(2)}$.

In the working of a quantum computer, entanglement has two opposite roles. On the one hand, entanglement between qubits is the key element in a quantum computation that distinguishes it from its classical counterpart [33]. On the other hand, when the computer and the environment become entangled, precious quantum information is lost. Usually, the latter effect is referred to as decoherence. In the literature, there are two different definitions of decoherence. In a strict

sense, decoherence is the decay in time of the coherences (off-diagonal elements of the reduced density matrix), while dissipation involves the exchange of energy with the environment and changes in populations (the diagonal terms of the density matrix). However, the word “decoherence” is also used in a broader sense involving changes in both diagonal and off-diagonal entries of the density matrix. In this paper we choose the latter use of the word. The reason is that from a quantum error correction perspective changes in diagonal and off-diagonal entries are “dual” to each other [1].

There is a simple heuristic explanation for error correction: Usually, noise is regarded as a local phenomenon, thus its damaging effect in the computer should be less pronounced if the information is delocalized among several qubits. This is precisely how classical error correction codes work. A simple example of the latter is a majority vote, where the information of a bit is copied into three physical bits, $0 \rightarrow 000$ and $1 \rightarrow 111$. If the probability of an error in a given qubit is ϵ , the probability of having two independent errors, and consequently a total information loss, is $\epsilon^2 \ll \epsilon$. Thus encoding increases the level of protection of the information.

It is tempting to start explaining QEC from this perspective. However, the no-cloning theorem [1] states that it is impossible to copy an unknown quantum state. The alternative approach is to use an entangled state involving two or more qubits to store the quantum information. This clearly delocalizes the information, but it is at odds with the intuitive notion that entangled states are in general more fragile to the effects of the environment (this intuition is driven by the quantum-to-classical transition due to decoherence, see Appendix D for a concrete example). Thus, delocalizing the information using entanglement does not alone solve the problem. It is possible to use unitary operations to transfer the entanglement between the qubits and the environment to a constant fresh supply of ancilla qubits [1,34]. However, it is more traditional in QEC to use the partial measurements of some ancilla qubits to reduce the quantum interference with the environment [1]. Measurements here must be understood as the projection of the state of one of the qubits (an ancilla) onto a certain basis or reference state. The outcome of this projection is a classical bit (“zero” or “one”) and is called a syndrome. The partial wave-packet reductions caused by syndrome extraction steer the long-time evolution of the quantum computer. Recently, it has been shown that the duration of the measurement is not fundamental to the QEC procedure [35]. In fact, this process can be quite long without jeopardizing the method.

A simple example illustrates how QEC works [1,9,10]. Suppose that we have an error model consisting of independent baths for each qubit which can cause only phase errors, and an initial qubit in the state $|\psi_0\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ that we want to protect. The three-qubit code provides the simplest error correction procedure for this problem. In Fig. 1, we define the encoding and/or decoding methods in a QEC cycle. At the end of a cycle, the probability of measuring the syndrome of a phase flip error in one of the three physical qubits is [39]

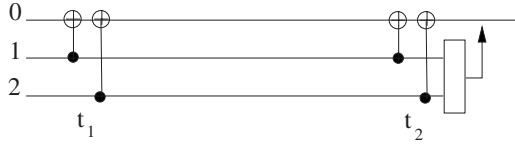


FIG. 1. A three-qubit quantum error correction (QEC) code [1,9,10,36,37]. The initial wave function, $|\psi_0\rangle \otimes (|\uparrow\rangle + |\downarrow\rangle)/2 \otimes (|\uparrow\rangle + |\downarrow\rangle)/2$, is encoded by two controlled-NOT (CNOT) gates, $R_{\text{CNOT}} = \sigma_i^- \sigma_j^+ \sigma_j^x + \sigma_i^+ \sigma_j^-$, into an entangled state $|\psi_{\text{encode}}\rangle = \alpha|\bar{\uparrow}\rangle + \beta|\bar{\downarrow}\rangle$ with $|\bar{\uparrow}\rangle = (|\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)/2$ and $|\bar{\downarrow}\rangle = (|\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)/2$. After some time, it is decoded by a second pair of CNOT gates. An error in $|\psi\rangle$ is identified by measuring the values of σ_2^x and σ_3^x (rectangle). The QEC cycle ends with the correction of a possible phase flip (arrow).

$$p_1 \approx 3\epsilon, \quad (11)$$

and the probability of the syndrome indicating no error in the logical qubit is

$$p_0 \approx 1 - p_1. \quad (12)$$

The residual decoherence that cannot be corrected by the QEC procedure is closely related to these probabilities. In the case of a cycle in which the syndrome indicates that one error occurred in any of the physical qubits, dephasing of the logical qubit is given by the reduction of the off-diagonal density matrix element [39],

$$\rho_{\bar{\uparrow}\bar{\downarrow}}^{(1)} \approx \alpha\beta^*(1 - 2\epsilon), \quad (13)$$

while for a cycle with a syndrome indicating no error, the dephasing is weaker,

$$\rho_{\bar{\uparrow}\bar{\downarrow}}^{(0)} \approx \alpha\beta^*(1 - 2\epsilon^3). \quad (14)$$

After N of these cycles, the probability of having m uncorrelated errors is

$$\mathcal{P}_m = \binom{N}{m} p_0^{N-m} p_1^m, \quad (15)$$

with an associated residual decoherence of

$$\rho_{\bar{\uparrow}\bar{\downarrow}}^{(m)} \approx \alpha\beta^*(1 - 2\epsilon^3)^{N-m}(1 - 2\epsilon)^m. \quad (16)$$

An elegant visualization of these events is given by a “syndrome history diagram” of Fig. 2 (see, for instance, Ref. [28] for a similar discussion). An ordered set of syndromes labels a particular evolution of the logical qubit. From the syndrome history one can find the most likely evolution and the associated residual decoherence. For our three-qubit code example, the most likely evolution is given by the mean value of m , $\bar{m} = Np_1$. Thus, the residual decoherence of the logical qubit is given by

$$\rho_{\bar{\uparrow}\bar{\downarrow}} \approx \alpha\beta^* e^{-6N\epsilon^2}. \quad (17)$$

Therefore, as long as the number of QEC cycles is $N \ll \epsilon^{-2}$, the probability of measuring the correct initial state of the logical qubit is very high. We can quantify the amount of

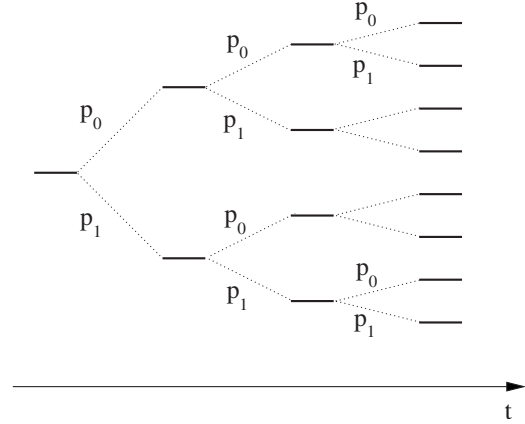


FIG. 2. A syndrome history diagram. Each solid line represents the evolution of a logical qubit. At the end of a QEC cycle, a phase-flip error is detected or not with probabilities p_1 and p_0 , respectively. A path provides the history of the logical qubit and is recorded as a sequence of syndromes.

information that is lost by calculating the von Neumann entropy $S = -\text{tr}(\rho \ln \rho)$,

$$\lim_{N \ll \epsilon^{-2}} S \approx 12N|\alpha|^2|\beta|^2\epsilon^2[1 - \ln(12N|\alpha|^2|\beta|^2\epsilon^2)], \quad (18)$$

$$\lim_{N \gg \epsilon^{-2}} S \approx -|\alpha|^2 \ln|\alpha|^2 - |\beta|^2 \ln|\beta|^2. \quad (19)$$

Note that the loss of information can be substantial if the number of cycles is so large that $N \gg \epsilon^{-2}$.

If the information needs to be protected for a long period of time, we must modify the protection scheme. The most straightforward approach is to consider a concatenated circuit where each qubit in Fig. 1 is a logical qubit itself and each gate is a logical gate, resulting in an effective reduction of p_1 . Layers and layers of protection can be added as needed [1,29]. A chief concern when applying this approach is whether the steps required in the addition of more qubits and operations do not actually increase the chance of errors (since they increase the combinatorial factors in the probability distribution). This question is addressed by fault-tolerant quantum computation theory [13,15–17,19,29], which has as its main result the so-called threshold theorem: If the “noise strength” ϵ is smaller than a certain critical value, then the introduction of an additional layer of concatenation improves the protection of the information.

A key ingredient in the derivation of the noise threshold is the assumption that a probabilistic structure similar to the one that we outlined above exists. Here rests the main concern of this paper. There are many physical situations where an environment can induce strong memory effects and spatial correlations among qubits. Hence, it may not be obvious how to define the “error probabilities” of a qubit. This hinders the traditional theory of QEC and threshold analysis, thus motivating a careful study of the dynamics of quantum computers protected by QEC.

II. ERROR MODELS AND QUANTUM CODES

The syndrome history used to describe the logical qubit history can be converted into a more formal description of the computer dynamics. In our discussion, we will assume an environment, H_0 , described by a free field theory with an ultraviolet cutoff Λ , a characteristic wave velocity v , and a dynamical exponent z . Although simple, a free field theory faithfully represents many physically relevant environments: The electromagnetic field, a phonon field, spin waves, a bosonic bath, or, more generally, any two-body direct interactions between qubits that was split by a Hubbard-Stratanovich field. In addition, we include in the Hamiltonian a term to account for the sequence of quantum gates performed on the qubits, $H_{QC}(t)$. Hence, the total Hamiltonian is

$$H(t) = H_0 + H_{QC}(t) + V. \tag{20}$$

The interaction term will be assumed to have the form of a vector coupling between qubits and the environment,

$$V = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda_{\alpha}}{2} f_{\alpha}(\mathbf{x}) \sigma^{\alpha}(\mathbf{x}), \tag{21}$$

where $\vec{\sigma}(\mathbf{x})$ are Pauli matrices for the qubit located at \mathbf{x} , λ_{α} are the coupling strengths, and $f_{\alpha}(\mathbf{x})$ are functions of the environment operators [38]. Since $[H_0, H_{QC}] = 0$, we adopt an interaction picture that follows not only the environment but also the evolution of the computer (see Appendix E). In this rotating frame, the evolution operator is

$$U(t, 0) = T_t e^{-i/\hbar \int_0^t dt' V(t')}. \tag{22}$$

The interaction $V(t)$ depends on the quantum code and its implementation. Nevertheless, there are two possible ways to keep the discussion code independent:

(i) In our previous work [39,40], we assumed that quantum gates were performed faster than the environment response time (which is of order the inverse of the ultraviolet cutoff frequency Λ). We call this approximation the “fast-gate” limit. For this case, we have the evolution of the computer between gates given by

$$V(t) = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda_{\alpha}}{2} f_{\alpha}(\mathbf{x}, t) \sigma^{\alpha}(\mathbf{x}), \tag{23}$$

with $f_{\alpha}(\mathbf{x}, t) = e^{iH_0 t/\hbar} f_{\alpha}(\mathbf{x}) e^{-iH_0 t/\hbar}$. Then, when a gate is performed the action on the qubit is instantaneous and the subsequent evolution is once again governed by Eq. (23).

(ii) A second possibility is to derive an upper bound on the effects of correlations. In order to do that, we must first discuss how slow gates, which are performed over time intervals larger than $\tau_c = 1/\Lambda$, change Eq. (23). Then, we can define an effective interaction V_{eff} that takes into account the slowness of the gates and serves as an upper bound to the exact (and code dependent) V . Clearly, the real experimental situation rests between the two limits (i) and (ii).

Before we begin a detailed description of how to handle case (ii), let us note that here the terminology “fast” and “slow” gates follows the QEC literature: Fast (slow) gates have a duration much shorter (longer) than τ_c . However, as will become clear later, the relevant time scale that appears

in the study of correlation effects is the period or duration of the error correction cycle, Δ . Thus, in that context, short (“fast”) or long (“slow”) dynamical effects will be naturally defined with respect to Δ , and not to τ_c .

Any quantum computer code is just a rotation in the Hilbert space of the qubits and can be described as a trajectory on $\mathbb{C}P^{2^N-1}$, where N is the total number of qubits. In the Schrödinger picture, the evolution is given by the natural action on S^{4^N-1} by $SU(2^N)$. The most general fault-tolerant quantum circuit is therefore defined by the Hamiltonian $H_{QC}(t) = \sum b_j(t) e_j$, where $\{e_j\}$ are the generators of the Lie algebra of $SU(2^N)$. The evolution operator associated with this Hamiltonian satisfies the integral equation

$$W(t, 0) = 1 - \frac{i}{\hbar} \int_0^t dt' H_{QC}(t') W(t', 0) = T_t e^{-i/\hbar \int_0^t dt' H_{QC}(t')}, \tag{24}$$

such that the computer state vector at time t is given by $|\psi(t)\rangle = W(t, 0)|\psi(0)\rangle$, where $|\psi(0)\rangle$ represents the initial state of the computer. Therefore, in the interaction picture, the interaction operator is given by

$$V(t) = W^\dagger(t) e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar} W(t) = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda_{\alpha}}{2} [e^{iH_0 t/\hbar} f_{\alpha}(\mathbf{x}) e^{-iH_0 t/\hbar}] W^\dagger(t) \sigma^{\alpha}(\mathbf{x}) W(t) \tag{25}$$

$$= \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda_{\alpha}}{2} f_{\alpha}(\mathbf{x}, t) W^\dagger(t) \sigma^{\alpha}(\mathbf{x}) W(t). \tag{26}$$

Since $W(t)$ is a $SU(2^N)$ matrix, then

$$G^{\alpha}(\mathbf{x}, t) = W^\dagger(t) \sigma^{\alpha}(\mathbf{x}) W(t) \tag{27}$$

is another matrix of $SU(2^N)$, and we can write

$$V(t) = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda_{\alpha}}{2} f_{\alpha}(\mathbf{x}, t) G^{\alpha}(\mathbf{x}, t). \tag{28}$$

Although the expression in Eq. (28) is general, it is not very instructive. Furthermore, it is very undesirable from an error correction standpoint: Since $G(t)$ is an arbitrary matrix of $SU(2^N)$, the $V(t)$ in Eq. (28) in principle generates a highly complex correlated error that is nevertheless first order in the coupling to the environment. The problem with the derivation of Eq. (28) is that it is too general since we assumed that arbitrary rotations are performed at each single step. However, one of the cornerstones of quantum computation is that such general rotations can be approximately decomposed into a series of elementary gates [1]. Hence, our strategy will be to specialize the calculation to these elementary gates and assume that general rotations can be implemented by a finite series of such gates which are well resolved in time.

A. Single-qubit operations

When only single-qubit operations are performed, we have

$$H_{\text{QC}}(t) = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} b_{\alpha}(\mathbf{x},t) \sigma^{\alpha}(\mathbf{x}). \quad (29)$$

In this case, $W(t)$ is the product of $SU(2)$ matrices acting in each qubit's Hilbert space. Thus, $G^{\alpha}(\mathbf{x},t)$ simplifies to

$$G_1^{\alpha}(\mathbf{x},t) = \begin{bmatrix} \rho_1 e^{-i\phi} & -\rho_2 e^{i\phi} \\ \rho_2 e^{-i\phi} & \rho_1 e^{i\phi} \end{bmatrix} \sigma^{\alpha}(\mathbf{x}) \begin{bmatrix} \rho_1 e^{i\phi} & \rho_2 e^{-i\phi} \\ -\rho_2 e^{-i\phi} & \rho_1 e^{-i\phi} \end{bmatrix}, \quad (30)$$

where $\rho_1^2 + \rho_2^2 = 1$ and $\{\rho_1, \rho_2, \phi, \varphi\}$ are functions of \mathbf{x} and t . The single-qubit rotations yield an expression of the form

$$G_1^{\alpha}(\mathbf{x},t) = \sum_{\beta=\{1,x,y,z\}} g^{\alpha\beta}(\mathbf{x},t) \sigma^{\beta}(\mathbf{x}) \quad (31)$$

for some $g^{\alpha\beta}(\mathbf{x},t)$. By decomposing the operators f_{α} and functions $g^{\alpha\beta}$ into their Fourier components, we can give a more formal meaning to ‘‘fast’’ and ‘‘slow’’ gates,

$$f_{\alpha}(\mathbf{x},t) g^{\alpha\beta}(\mathbf{x},t) = \sum_{|\omega_1| < \Lambda, \omega_2} e^{i(\omega_1 + \omega_2)t} f_{\alpha}(\mathbf{x},\omega_1) g^{\alpha\beta}(\mathbf{x},\omega_2). \quad (32)$$

Hence, if we define $\nu = \omega_1 + \omega_2$, we can rewrite the perturbation as

$$V = \sum_{\beta} \left[\sum_{\nu} e^{i\nu t} \left(\sum_{\omega_2} \sum_{\alpha} f_{\alpha}(\nu - \omega_2) g^{\alpha\beta}(\omega_2) \right) \right] \sigma_{\beta}. \quad (33)$$

In the limit of fast gates, $|\omega_2| > \Lambda$, f and g are not convolved, since they have distinct frequency domains. Therefore, the noise operators f_{α} are unaltered by the rotation. However, if g has a significant weight at frequencies smaller than Λ (slow gates), one must convolve f with g , yielding a substantially different noise operator.

B. Two-qubit operations

The general Hamiltonian for two-qubit gates is of the form

$$H_{\text{QC}}(t) = \sum_{\mathbf{x}, \mathbf{y}} \sum_{\alpha, \beta=\{x,y,z\}} J^{\alpha\beta}(\mathbf{x}, \mathbf{y}, t) \sigma^{\alpha}(\mathbf{x}) \sigma^{\beta}(\mathbf{y}). \quad (34)$$

However, one can also generate a full set of gates using instead a single type of interaction,

$$H_{\text{QC}}(t) = \sum_{\mathbf{x}, \mathbf{y}} J(\mathbf{x}, \mathbf{y}, t) \sigma^a(\mathbf{x}) \sigma^b(\mathbf{y}), \quad (35)$$

where a and b are fixed for each gate (\mathbf{x}, \mathbf{y}) . In order to see that this is sufficient we can, for instance, set $a=b=z$. This generates the liquid NMR Hamiltonian [41], where the Ising interaction, Eq. (35), and single-qubit rotations can be used to generate a control σ^z gate.

We keep a and b arbitrary. However, for the sake of simplicity, we assume that only operations between disjoint pairs are allowed; that is, if $J(\mathbf{x}, \mathbf{y}_1, t) \neq 0$, then $J(\mathbf{x}, \mathbf{y}_2, t) = 0$ for all $\mathbf{y}_2 \neq \mathbf{y}_1$. It is then straightforward to write down $W(t)$ in a

compact form: The time ordering [Eq. (24)] is automatically taken care of by the sequence of gates, while for a gate involving qubits \mathbf{x} and \mathbf{y} the contribution to $W(t)$ is

$$W(\mathbf{x}, \mathbf{y}, t) = \cos[\theta(\mathbf{x}, \mathbf{y}, t)] + i \sin[\theta(\mathbf{x}, \mathbf{y}, t)] \sigma^a(\mathbf{x}) \sigma^b(\mathbf{y}), \quad (36)$$

where $\theta(\mathbf{x}, \mathbf{y}, t) = \int_0^t dt' J(\mathbf{x}, \mathbf{y}, t')$. Hence, a two-qubit rotation yields

$$G_2^{\alpha}(\mathbf{x}, t) = \sin[2\theta(\mathbf{x}, \mathbf{y}, t)] \epsilon^{a\alpha\gamma} \sigma^{\gamma}(\mathbf{x}) \sigma^b(\mathbf{y}) + \cos[2\theta(\mathbf{x}, \mathbf{y}, t)] (1 - \delta_{a,\alpha}) \sigma^{\alpha}(\mathbf{x}) + \delta_{a,\alpha} \sigma^{\alpha}(\mathbf{x}), \quad (37)$$

where $\epsilon^{a\alpha\gamma}$ is the usual antisymmetric tensor.

The first term on the right-hand side (rhs) of Eq. (37) tells us that the two-qubit gate can propagate the error from the qubit at \mathbf{x} to the qubit at position \mathbf{y} . However, it also tells us that it is possible to choose a particular gate where this propagation does not happen (by choosing $a=\alpha$, for instance). Unfortunately, propagating errors in the quantum circuit is in general unavoidable (since the only gate that commutes with all Pauli operators is the identity).

The second and third terms on the rhs of Eq. (37) are much less dramatic. They simply describe a local noise that is not propagated by the gate.

C. Upper bounds for the evolution

In Eqs. (31) and (37), we showed that one- and two-qubit gates can introduce what is seemingly a very complicated noise structure. The expressions depend on how the gates are implemented, thus hiding a general assessment. We can advance the discussion by recalling that W is always a unitary matrix. Hence, the coefficients in Eqs. (31) and (37) have modulus equal or smaller than unity. A suitable upper bound on the effects of slow gates is then provided by setting all these coefficients equal to 1. Thus, the operators expressed in Eqs. (31) and (37) gain the upper bounds

$$\tilde{G}_1^{\alpha}(\mathbf{x}) = \sum_{\beta=\{x,y,z\}} \sigma^{\beta}(\mathbf{x}), \quad (38)$$

$$\tilde{G}_2^{\alpha}(\mathbf{x}) = \sigma^{\alpha}(\mathbf{x}) + \epsilon^{a\alpha\gamma} \sigma^{\gamma}(\mathbf{x}) \sigma^b(\mathbf{y}). \quad (39)$$

\tilde{G}_2^{α} still looks troublesome, since it tells us that an error in qubit \mathbf{x} is propagated to \mathbf{y} . However, this is not a problem of the finite gate time operation, since an instantaneous and perfect gate will also propagate the error in a similar fashion. In order to obtain an upper bound for the effects introduced by the two-qubit gates, we precisely follow this fact. We consider that all the qubit components are exposed to all the noise channels all the time. Thus, we replace Eq. (39) by $G_2^{\alpha}(\mathbf{x}) = \sum_{\beta=\{x,y,z\}} \sigma^{\beta}(\mathbf{x})$ and assume that two-qubit gates are performed instantaneously. In summary, we reduce the problem of finite time operation of the two-qubit gate to the problem of a noisier qubit environment and propagating errors in the quantum code by perfect gates. Now we can rely on the theory of fault tolerance [1,29], and simply assume that the error propagation is handled by the quantum code.

The final conclusion is that an upper bound estimate on the effects of slow gates is obtained by the interaction Hamiltonian

$$V_{\text{eff}}(t) = \sum_{\mathbf{x}} \sum_{\alpha=\{x,y,z\}} \frac{\lambda}{2} f_{\text{eff}}(\mathbf{x}, t) \sigma^{\alpha}(\mathbf{x}, t), \quad (40)$$

where

$$f_{\text{eff}}(\mathbf{x}, t) = \frac{1}{\lambda^2} \left(\sum_{\beta=\{x,y,z\}} \lambda_{\beta} f_{\beta}(\mathbf{x}, t) \right) \quad (41)$$

and $\lambda = \sqrt{\sum_{\beta=\{x,y,z\}} \lambda_{\beta}^2}$ is the new coupling parameter. Although this is a brutal approximation, it will be sufficient for our discussion. As we will argue later, for the purpose of determining the effect of long-wavelength correlations on the threshold theorem, the only relevant aspect of the f_{α} is their scaling dimension. Since $\dim f_{\text{eff}}$ is in general equal to $\min(\dim f_{\alpha})$, it is sufficient to use Eq. (40) as the worst case scenario.

Thus, in both limiting cases, fast and slow gates, we arrive at the *same functional form* for the effective interaction. Hence, both cases can be handled simultaneously, and we proceed to the analysis of QEC in the presence of this interaction. In order to simplify the notation, we hereafter drop the subscript “eff” from the slow-gate operators.

III. QUANTUM ERROR CORRECTION IN CORRELATED ENVIRONMENTS

A QEC code is defined as the combination of encoding, decoding, and recovery operations. Since we were able to make our analysis code independent, the unitary component of the QEC protocol is described by $U(\Delta, 0)$, Eq. (22), with the appropriate $V(t)$ discussed in Sec. II. The final ingredient in standard QEC is just the syndrome extraction P , which is a projective measurement.

In Ref. [39] it was demonstrated how to define P and its effects on U for stabilizer error correction codes. It is important to remark that an error which keeps the computer in the logical Hilbert space can never be corrected by QEC. This is simply a statement that for the general assumptions we make, the problem of protecting quantum information never satisfies the second criterion of Laflamme-Knill [12] for perfect QEC. In simple terms, the criteria states that all allowed errors must always take the logical one and logical zero to orthogonal states [Eq. (20) of Ref. [12]]. By construction, these errors are high-order events in the coupling with the environment. Nevertheless, as we already know (Sec. I A), this fact is not enough to ensure that such errors will not be relevant at long times. One of our goals is to find out when it is appropriate to safely neglect such uncorrectable errors in the presence of correlated environments.

In hindsight, it is not hard to understand the benefits of QEC. Thus, for the sake of readability, we present first a qualitative argument that captures the overall discussion.

As we defined in the introduction, there are two quantities that we are interested in calculating: (i) The probability of a

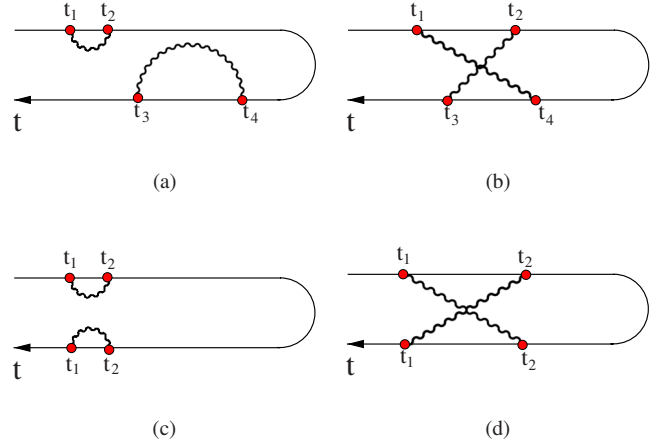


FIG. 3. (Color online) Graphical representation of a few fourth-order terms in a “time-loop” expansion for either the probability of a given evolution or the reduced density matrix (spatial dimensions are suppressed for clarity). Points of interaction with the bath (circles) are connected by propagation of the environmental modes (wiggly lines). In the top diagrams, the time integrals are unconstrained, as would be the case for unitary evolution. In the bottom diagrams, the detection of an error by a QEC protocol forces the interactions with the bath to occur at the same times on both the forward and backward legs in order that U and U^\dagger correspond to the same syndrome. This additional constraint introduced by QEC is crucial in the long-time behavior.

given evolution, and (ii) the reduced density matrix of the computer. Both quantities are written as a double series in the coupling with the environment. On the one hand, the initial ket of computer and the environment, $|\Psi\rangle$, evolves in the time interval $[0, t]$ by the time-ordered series $U(t)$. On the other hand, the bra $\langle\Psi|$ evolves in time with the anti-time-ordered series U^\dagger . It is only a subset of each series that enters in the evaluation of either the probability or the reduced density matrix, because of the measurements present in the traditional formulation of QEC. Hence, it is usually a nontrivial task to calculate the necessary expectation values.

Because we are dealing with a double series, it is natural to use a formalism analogous to a time-loop expansion [42]. There are six (interrelated) Green functions in such an expansion: The usual advanced and retarded functions for the time-ordered series; the advanced and retarded functions for the anti-time-ordered series; and the lesser and greater functions, which contract a term from the time-ordered series with another one from the anti-time-ordered series. This formalism is often referred to as the Schwinger-Keldysh approach [43,44]. It is usually represented graphically by a double contour in time (see Fig. 3). The upper leg stands for the time-ordered evolution for the time interval $[0, t]$, while the lower leg stands for the anti-time-ordered evolution in the reversed interval $[t, 0]$.

Let us for the moment assume that a short-time expansion is valid and focus on a single qubit. Then, the evolution operator for that particular qubit within a QEC cycle is given by

$$U_1(\Delta, 0) \approx 1 - \frac{i}{\hbar} \int_0^\Delta dt \sum_{\alpha=\{x,y,z\}} \frac{\lambda_\alpha}{2} f_\alpha(\mathbf{x}, t) \sigma^\alpha(\mathbf{x}, t) - \frac{1}{\hbar^2} \int_0^\Delta dt \int_0^t dt' \sum_{\alpha=\{x,y,z\}} \frac{\lambda_\alpha \lambda_\beta}{4} f_\alpha(\mathbf{x}, t) f_\beta(\mathbf{x}, t') \sigma^\alpha(\mathbf{x}, t) \sigma^\beta(\mathbf{x}, t') + O(\lambda^3). \quad (42)$$

In Fig. 3 we represent graphically a few terms of order λ^4 . All of these terms are the product of a second-order term from U_1 and a second-order term from U_1^\dagger [see Eq. (42)]. Hence, they correspond to two “errors” in the qubit evolution and involve the expectation value

$$\langle \Psi | f_\alpha^\dagger(\mathbf{x}, t) f_\beta^\dagger(\mathbf{x}, t') f_\alpha(\mathbf{x}, t'') f_\beta(\mathbf{x}, t''') | \Psi \rangle. \quad (43)$$

Using Wick’s theorem, we can immediately write (43) as a product of the noninteracting Green functions. Each possible set of contractions leads to the different “diagrams” in Fig. 3.

We usually do not know when an “error” occurs; hence, each Green function is accompanied in the series by a double integral in time. This is precisely the case in an unprotected computer’s evolution or inside a QEC cycle [see Figs. 3(a) and 3(b)]. However, a dramatic change happens in a Green function between terms for different cycles. When the syndrome shows that a particular error occurred in a certain QEC cycle, we can rewrite Eq. (43) to reflect this knowledge,

$$\langle \Psi | f_\alpha^\dagger(\mathbf{x}, t) f_\beta^\dagger(\mathbf{x}, t') f_\alpha(\mathbf{x}, t + \delta t) f_\beta(\mathbf{x}, t' + \delta t') | \Psi \rangle, \quad (44)$$

where δt and $\delta t'$ are time variables with range smaller than the QEC period. After integrating the “high frequency” part (the δt and $\delta t'$ variables), we end up reducing Eq. (44) to

$$\langle \Psi | f_\alpha^\dagger(\mathbf{x}, t) f_\beta^\dagger(\mathbf{x}, t') f_\alpha(\mathbf{x}, t) f_\beta(\mathbf{x}, t') | \Psi \rangle \quad (45)$$

with t and t' representing a coarse-grained time scale of the order of the QEC period [see Figs. 3(c) and 3(d)]. Therefore, although we are considering terms of the same order in λ , the number of “time integrals” in the coarse-grained scale (low frequencies) is one-half that in the original microscopic calculation (high frequencies).

The simple dimensional analysis of Sec. I A tells us now that QEC has changed the criteria for the stability of the perturbation series at long times. As we demonstrate now, it is less stringent than the naive expectation.

A. Quantum evolution steered by QEC

It is reasonable to assume that at the beginning of the computation the computer’s state vector, $|\psi_0\rangle$, and the environment’s, $|\varphi_0\rangle$, are not entangled,

$$|\Psi(t=0)\rangle = |\psi_0\rangle \otimes |\varphi_0\rangle. \quad (46)$$

In a realistic situation, $|\psi_0\rangle$ would have some initialization error and be entangled with the environment to some degree (both of which would yield errors in $|\psi\rangle$). However, here we neglect these effects in order to keep the discussion focused.

Just as in the case of the three-qubit code, by the end of a QEC cycle the computer will have evolved according to the

unitary operator $U(\Delta, 0)$. Then, the syndrome is extracted and the computer wave function is projected,

$$P_m U(\Delta, 0) |\Psi(0)\rangle, \quad (47)$$

where m corresponds to a particular syndrome, with $\sum_m P_m = I$ and $P_m^2 = P_m$. In the case of many logical qubits evolving together, then m denotes the set of all the syndromes extracted at time Δ . The last step in the code is the appropriate recovery operation, R_m , depending on the syndrome outcome,

$$|\Psi(\Delta)\rangle = R_m(\Delta + \delta, \Delta) P_m U(\Delta, 0) |\Psi(0)\rangle. \quad (48)$$

Since in a fault-tolerant error correction scheme the information is never decoded (in contrast to the three-qubit code discussed above), the quantum information always remains protected. Therefore, we can deal with our two limiting cases (slow and fast gates) in two different ways. In the case of a slow-gate recovery, we formally include it as the initial step of the next QEC period. Conversely, in the case of fast gates, we assume that the recovery is performed flawlessly in a very short time scale after the projection. For the sake of clarity, we choose the latter below. We emphasize that this does not restrict our discussion, since it is known that the time of recovery is irrelevant to the error correction. In fact, it can be postponed all the way to the end of the calculation [35].

B. Probability of a syndrome history and the loss of information

The first quantity to discuss is the probability of measuring a particular syndrome at the end of the first QEC step,

$$P_m = \langle \Psi(0) | U^\dagger(\Delta, 0) P_m U(\Delta, 0) | \Psi(0) \rangle. \quad (49)$$

The corresponding reduced density matrix is

$$\begin{aligned} \rho_{r,s}^m(\Delta) &= \frac{\text{tr}_e[\langle \vec{r} | P_m U(\Delta, 0) | \Psi(0) \rangle \langle \Psi(0) | U^\dagger(\Delta, 0) P_m | \vec{s} \rangle]}{\langle \Psi(0) | U^\dagger(\Delta, 0) P_m U(\Delta, 0) | \Psi(0) \rangle} \\ &= \frac{\langle \varphi_0 | [\langle \psi_0 | U^\dagger(\Delta, 0) P_m | \vec{s} \rangle \langle \vec{r} | P_m U(\Delta, 0) | \psi_0 \rangle] | \varphi_0 \rangle}{\langle \varphi_0 | \langle \psi_0 | U^\dagger(\Delta, 0) P_m U(\Delta, 0) | \psi_0 \rangle | \varphi_0 \rangle}, \end{aligned} \quad (50)$$

where \vec{r} and \vec{s} denote states in the computer Hilbert space and tr_e is the trace over the environment Hilbert space. It is possible to quantify how much information was leaked to the environment by calculating the von Neumann entropy

$$S(\Delta) = -\text{tr}_c[\rho^m(\Delta) \ln |\rho^m(\Delta)|], \quad (51)$$

where tr_c is the trace over the computer Hilbert space.

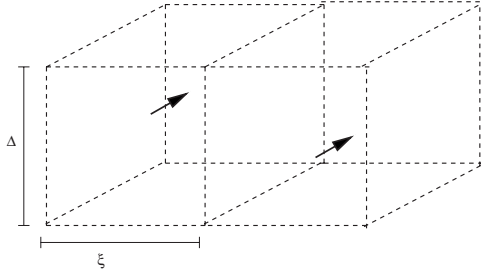


FIG. 4. Two neighboring hypercubes in space time, each one containing a qubit.

In Eqs. (49) and (50), one clearly sees the important role played by the projection operators in the quantum evolution steered by QEC. The careful construction of the encoded states combined with the measurement (syndromes) reduces the quantum interference between different history paths of the computer. By partially collapsing the wave function of the computer, this traditional form of QEC reduces decoherence.

Equations (49) and (50) define the local components of the noise. When spatial correlation between qubits can be ignored, they are related to the stochastic probabilities and density matrix discussed in Sec. I B [see Eqs. (15) and (16)].

The generalization to a sequence of QEC cycles is straightforward [39],

$$Y_{\mathbf{w}} = u_{w_N}(N\Delta, (N-1)\Delta) \cdots u_{w_1}(\Delta, 0), \quad (52)$$

where \mathbf{w} is the particular history of syndromes for all the qubits and

$$u_{w_j}(j\Delta, (j-1)\Delta) = R_{w_j}(j(\Delta + \delta), j\Delta) P_{w_j} U(j\Delta, (j-1)\Delta), \quad (53)$$

is the QEC evolution after each cycle. Each history comes with the associated probability

$$\mathcal{P}(Y_{\mathbf{w}}) = \langle \varphi_0 | \langle \psi_0 | Y_{\mathbf{w}}^\dagger Y_{\mathbf{w}} | \psi_0 \rangle | \varphi_0 \rangle. \quad (54)$$

Finally, there is always some residual decoherence which can be found from the reduced density matrix

$$\rho_{\vec{r}, \vec{s}}(Y_{\mathbf{w}}) = \frac{\langle \varphi_0 | [\langle \psi_0 | Y_{\mathbf{w}}^\dagger | \vec{s} \rangle \langle \vec{r} | Y_{\mathbf{w}} | \psi_0 \rangle] | \varphi_0 \rangle}{\langle \varphi_0 | \langle \psi_0 | Y_{\mathbf{w}}^\dagger Y_{\mathbf{w}} | \psi_0 \rangle | \varphi_0 \rangle}, \quad (55)$$

with \vec{r} and \vec{s} being elements of the logical subspace. This in turn yields the entropy

$$S(Y_{\mathbf{w}}) = -\text{tr}_c[\rho(Y_{\mathbf{w}}) \ln[\rho(Y_{\mathbf{w}})]]. \quad (56)$$

In the following, we will show for Eqs. (54) and (55) how to separate the effect of correlations between different QEC cycles from the contributions due to the local component of the noise, as defined by Eqs. (49) and (50).

IV. PERTURBATION THEORY AND THE HYPERCUBE ASSUMPTION

There is one additional issue that we must deal with before we can move forward. In principle, even the first-order

term in Eq. (42) is already beyond the QEC approach that has been outlined so far. The reason is that when calculating \mathcal{P} or ρ we generate pair contractions of the type $\langle f_\alpha(\mathbf{x}, t) f_\alpha(\mathbf{y}, t') \rangle$. Therefore, the probability of finding an error at a given qubit is conditional on what happens with all other qubits. This automatically hinders the simple probabilistic interpretation of QEC that we used in Sec. I B.

The fact that we do not want to deal with such conditional probabilities leads us to the single most important simplifying hypothesis of our work: We assume that the qubits are separated by a minimum distance

$$\xi = (v\Delta)^{1/z}, \quad (57)$$

where v is the excitation velocity and z is the dynamical exponent of the theory describing the environment. Hence, for all $\mathbf{x} \neq \mathbf{y}$ and $|t-t'| < \Delta$, we have $\langle f_\alpha(\mathbf{x}, t) f_\alpha(\mathbf{y}, t') \rangle \approx 0$. It is then possible to assign a probability for the short-time evolution of each qubit independently of all others.

To further organize the analysis we order the qubits in a D -dimensional array that defines hypercubes of volume $\Delta \times \xi^D$ (see Fig. 4). In summary, for times smaller than Δ , each qubit has a dynamics independent from the other qubits, hence resembling a quantum impurity problem. However, for time scales larger than Δ , spatial correlations among them are present, thus making the problem similar to a spin lattice.

Ideally, we would like to decompose the evolution operator in interhypercube and intrahypercube components,

$$U(\Delta, 0) = U_{<}(\Delta, 0) U_{>}(\Delta, 0), \quad (58)$$

where the less than symbol labels frequencies smaller than Δ^{-1} and the greater than symbol labels frequencies in the interval $[\Delta^{-1}, \Lambda]$. Whenever this is possible, we can integrate the intrahypercube part in order to define a “local” evolution and, consequently, a local error probability. There are simple noise models where this can be done exactly [39], however, in general, this separation is only possible in a perturbative expansion. Keeping just a few terms in perturbation theory is not always adequate, and we must try to find ways to improve it.

A. Perturbation theory improved by RG

Our objective in this section is to define an effective evolution operator that can reasonably describe the evolution of the qubit within each hypercube. All terms consistent with the same syndrome and having the same leading long-time properties should be included. Within a hypercube, the environment induces interaction of a qubit only with itself; communication between qubits at longer times is treated in the next section.

We use the renormalization group (RG) [45] to sum the most relevant families of terms in the perturbation series. In order to improve the lowest-order terms in the perturbation theory through RG, we need to introduce the next higher-order terms in the perturbation series. However, as we discussed previously, we are not interested in the full unitary evolution, but rather the projected terms obtained after the extraction of the syndrome. Therefore, in order to apply RG to the first-order term, we need to consider

$$\begin{aligned}
v_\alpha(\mathbf{x}_1, \lambda_\alpha) \approx & -\frac{i}{2\hbar} \lambda_\alpha \int_0^\Delta dt f_\alpha(\mathbf{x}_1, t) - \frac{1}{8\hbar^2} |\epsilon_{\alpha\beta\gamma}| \lambda_\beta \lambda_\gamma \sigma_\alpha(\Delta) T_t \int_0^\Delta dt_1 dt_2 f_\beta(\mathbf{x}_1, t_1) f_\gamma(\mathbf{x}_1, t_2) \sigma_\beta(t_1) \sigma_\gamma(t_2) \\
& + \frac{i}{48\hbar^3} \sum_\beta \lambda_\alpha \lambda_\beta^2 \sigma_\alpha(\Delta) T_t \int_0^\Delta dt_1 dt_2 dt_3 f_\alpha(\mathbf{x}_1, t_1) f_\beta(\mathbf{x}_1, t_2) f_\beta(\mathbf{x}_1, t_3) \sigma_\alpha(t_1) \sigma_\beta(t_2) \sigma_\beta(t_3), \quad (59)
\end{aligned}$$

where $\epsilon_{\alpha\beta\gamma}$ is the antisymmetric tensor [46]. There is only one spatial index in (59) because of the hypercube assumption: We have included only terms in which contraction of the f 's yields a nonzero value, as these will contribute to the effective short-time evolution. At long times, connections between the qubits are, of course, essential, and this is treated in the next section.

The RG is naturally implemented in the case of ohmic baths (which leads to logarithmic singularities). However, suitable generalizations can be defined by dimensional regularization or by summing series in the expansion. Thus, in general, it is possible to write the following β function for $v_{\mathbf{x}_1}^\alpha$:

$$\frac{d\lambda_\alpha}{d\ell} = g_{\beta\gamma}(\ell) \lambda_\beta \lambda_\gamma + \sum_\beta h_{\alpha\beta}(\ell) \lambda_\alpha \lambda_\beta^2, \quad (60)$$

where g and h are functions specific to a particular environment, $\ell = \Lambda/\Lambda'$, and Λ' is the reduced (i.e., rescaled) cutoff frequency. By integrating the β function from the bare cutoff, Λ , to Δ^{-1} , we are summing the most relevant components of the noise inside a hypercube. If the renormalized value of the running coupling at frequency Δ^{-1} , λ^* , is still a small number, then it is a good approximation to consider

$$v_\alpha(\mathbf{x}_1, \lambda_\alpha^*) \approx \frac{-i\lambda_\alpha^*}{2\hbar} \int_0^\Delta dt f_\alpha(\mathbf{x}_1, t) \quad (61)$$

as the evolution operator of the qubit at position \mathbf{x}_1 which was diagnosed with an error α by the QEC procedure.

We illustrate the renormalization group procedure with two simple examples of ohmic baths: (i) Marginally relevant and (ii) marginally irrelevant couplings.

1. k -channel Kondo problem

The first example is a qubit exposed to a bosonic bath that is modeled by a $SU(2)_k$ Kac-Moody algebra—the bosonized Hamiltonian of a k -channel Kondo problem. Here we closely follow the work of Affleck and Ludwig (see Appendix B of Ref. [47]). We define chiral bosonic currents $:J_L$: Obeying the operator product expansion (OPE)

$$:J_L^a(t)::J_L^b(t'): \rightarrow \frac{f^{abc} :J_L^c(t):}{v(t-t')} - \frac{k\delta^{ab}}{2v^2(t-t')^2}, \quad (62)$$

where f^{abc} are the group structure constants and v is the velocity of excitations. In the interaction picture, the qubit couples to the currents by the usual Kondo interaction, yielding an evolution operator (or, equivalently, a scattering matrix) of the form

$$U = T_t e^{-i\lambda v/2\hbar \int_{-\infty}^{\infty} dt :J_L(t) \cdot \vec{\sigma}}. \quad (63)$$

Following our general discussion, we expand the evolution operator to lowest order in the coupling,

$$\begin{aligned}
U \approx & 1 - \frac{i\lambda v}{2\hbar} \int_{-\infty}^{\infty} dt :J_L(t) \cdot \vec{\sigma} - \left(\frac{\lambda v}{2\hbar}\right)^2 \sum_{a,b} \int_{-\infty}^{\infty} dt \int_{-\infty}^t dt' :J_L^a(t)::J_L^b(t'):\sigma^a \sigma^b \\
& + i \left(\frac{\lambda v}{2\hbar}\right)^3 \sum_{a,b,c} \int_{-\infty}^{\infty} dt \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' :J_L^a(t)::J_L^b(t')::J_L^c(t''):\sigma^a \sigma^b \sigma^c. \quad (64)
\end{aligned}$$

Due to the QEC evolution, only some of these terms are kept after the syndrome is extracted [see Eq. (59)]. For clarity, let us assume that we know from the syndrome that a phase flip has occurred. Hence, we must truncate the evolution operator to reflect this fact and apply the recovery operation (in this case multiply by σ^z), yielding

$$\begin{aligned}
v_z \approx & -\frac{i\lambda v}{2\hbar} \int_{-\infty}^{\infty} dt :J_L^z(t) : - i \left(\frac{\lambda v}{2\hbar}\right)^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^t dt' [:J_L^x(t)::J_L^y(t') : - :J_L^y(t)::J_L^x(t') :] \\
& + i \left(\frac{\lambda v}{2\hbar}\right)^3 \sum_a \int_{-\infty}^{\infty} dt \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' [:J_L^a(t)::J_L^a(t')::J_L^z(t'') : + :J_L^z(t)::J_L^a(t')::J_L^a(t'') : - :J_L^a(t)::J_L^z(t')::J_L^a(t'') :]. \quad (65)
\end{aligned}$$

Now, we integrate over a small frequency shell $[\Lambda - \delta\Lambda, \Lambda]$ and invoke the OPE. The result is a renormalization of the coupling λ by an infinitesimal composed of quadratic and cubic terms,

$$\frac{d\lambda}{d\ell} = \lambda^2 - \frac{k}{2}\lambda^3. \quad (66)$$

The resulting running coupling $\lambda(\ell)$ can be used to improve the results of our bare perturbation theory. For that purpose, we integrate the β function from the bare cutoff until Δ^{-1} . For the case of a small number of channels, we obtain a renormalized coupling of the form

$$\lambda^* \approx \frac{\lambda}{1 - \lambda \ln|\Lambda\Delta|}. \quad (67)$$

Although the RG flow goes toward the strong coupling limit, we do not integrate the β function all the way to zero frequency. Thus, if the renormalized coupling λ^* is still a small parameter, it replaces λ leading to the first-order renormalized evolution

$$v_z \approx -\frac{i\lambda^* v}{2\hbar} \int_{-\infty}^{\infty} dt:J_L^z(t):\sigma^z. \quad (68)$$

2. Quantum frustrated system

Correlations are not necessarily malignant to the computer's behavior. This is illustrated by our second example: A quantum frustrated environment [48–50]. Consider the case of three independent Abelian ohmic baths coupled as in Eq. (63), but with the OPE

$$:J_L^a(t)::J_L^b(t'): \rightarrow -\frac{\delta^{ab}}{2v^2(t-t')^2}. \quad (69)$$

Following precisely the same methodology of the previous example, we obtain the β function

$$\frac{d\lambda}{d\ell} = -\frac{1}{2}\lambda^3, \quad (70)$$

which leads to the renormalized coupling

$$\lambda^* \approx \frac{\lambda}{\sqrt{1 + 2\lambda^2 \ln|\Lambda\Delta|}}. \quad (71)$$

A quantum frustrated system has the remarkable property of asymptotic freedom. Hence, even very large bare couplings flow toward a perturbative regime. The physical reason behind this is the lack of a pointer basis [51], thus effectively decoupling the qubit from its surroundings [49]. This phenomena can also be understood as self-inflicted π -pulse decoupling working at the cutoff frequency Λ [7,52].

If the three coupling constants have different bare values, then the flow stops at some finite frequency since two of the couplings will flow to zero before the third. In other words, there will be a pointer basis. In a quantum computer protected by QEC, however, we are effectively stopping the flow at a finite frequency. Hence, the effect described in the preceding paragraph is relevant even for large anisotropic couplings.

B. Probability of a faulty path

Now that we have obtained a reasonable approximation to the evolution operator at each QEC step, we can turn to the problem of evaluating how much protection QEC yields at long times. The simplest quantity to calculate is the probability of finding a particular history of syndromes, Eq. (54). Using Eq. (61) and the known commutation relations of the f_α operators, we in general can write that

$$Y_w^\dagger Y_w = v_{w_N}^2 (N\Delta, (N-1)\Delta) \cdots v_{w_1}^2 (\Delta, 0), \quad (72)$$

and define

$$v_w^2(\Delta, 0) \approx \sum_{ij} \frac{\lambda_{\alpha_i}^* \lambda_{\alpha_j}^*}{4\hbar^2} \int_0^\Delta dt_1 dt_2 f_{\alpha_i}^\dagger(\mathbf{x}_i, t_1) f_{\alpha_j}(\mathbf{x}_j, t_2). \quad (73)$$

We now can evoke Wick's theorem once again to separate the intrahypercube and interhypercube contributions to the probability: The quantum average $\mathcal{P}(Y_w) \approx \langle \varphi_0 | Y_w^\dagger Y_w | \varphi_0 \rangle$ can be written as a sum of all possible pair contractions. It is convenient to separate the sum into two distinct parts.

First, the sum of all pair contractions in the same hypercube gives the stochastic error probability of a qubit, that we defined in Eq. (49), namely,

$$\epsilon_\alpha = \langle \varphi_0 | v_\alpha^2(\mathbf{x}_1, \lambda_\alpha^*) | \varphi_0 \rangle = \left(\frac{\lambda_\alpha^*}{2\hbar} \right)^2 \int_0^\Delta dt_1 dt_2 \langle f_\alpha^\dagger(\mathbf{x}, t_1) f_\alpha(\mathbf{x}, t_2) \rangle, \quad (74)$$

where we used again that for $|\mathbf{x} - \mathbf{y}| > \xi$ and $|t_1 - t_2| < \Delta$, we have $\langle f_\alpha^\dagger(\mathbf{x}, t_1) f_\alpha(\mathbf{y}, t_2) \rangle \approx 0$. Note that when we calculated λ^* we already summed intrahypercube pair contractions; however, these were contractions on the same Keldysh branch [see Fig. 3(a)] and therefore are related to the wave-function amplitude. Equation (74) corresponds to pair contractions between two distinct Keldysh branches [see Fig. 3(b)], hence it gives the probability of that evolution. With this two-step procedure, we sum up the most relevant contributions to the probability within a hypercube.

Second, we sum contractions between hypercubes. For each possible syndrome outcome we define the operators

$$F_0(\mathbf{x}, 0) = 1 - \frac{\sum (\lambda_\alpha^* \Delta / 2\hbar)^2}{1 - \sum_\alpha \epsilon_\alpha} :|f_\alpha(\mathbf{x}, 0)|^2: \quad (75)$$

and

$$F_\alpha(\mathbf{x}, 0) = \frac{1}{\epsilon_\alpha} \left(\frac{\lambda_\alpha^* \Delta}{2\hbar} \right)^2 :|f_\alpha(\mathbf{x}, 0)|^2:, \quad (76)$$

where $::$ stands for normal ordering with respect to the environment ground state (see Appendix F). We use these operators to express the remaining pair contractions of each hypercube in the probabilities, namely,

$$v_0^2(\mathbf{x}, \Delta, 0) \approx \left(1 - \sum_\alpha \epsilon_\alpha \right) F_0(\mathbf{x}, 0) \quad (77)$$

and

$$v_\alpha^2(\mathbf{x}, \Delta, 0) \approx \epsilon_\alpha [1 + F_\alpha(\mathbf{x}, 0)]. \quad (78)$$

Equations (77) and (78) are the final ingredients needed to evaluate the probability of a particular history of syndromes, Eq. (54). The remarkable aspect of these equations is that they provide a very elegant reorganization of the perturbation series. They were tailored to separate the local contribution, ϵ_α , from the long-distance, long-time components of the noise, F_α . The high-frequency part gives rise to the stochastic noise that is well discussed in the QEC literature. We rewrote the rest of the series taking into account the unusual nonunitary driven dynamics of QEC. The only remaining issue is to evaluate the stability of the perturbation expansion in the renormalized coupling λ^* .

In Sec. I A we discussed how the scaling dimension of an operator is important when studying a perturbative expansion. The same argument holds when evaluating the protection yielded by QEC in a correlated environment. If the scaling dimension of f_α is δ_α , then $\dim F_\alpha = 2\delta_\alpha$ (see Appendix G). Hence, the original criterion for the validity of the perturbative expansion in λ , $D+z-\delta_\alpha < 0$, becomes

$$D + z - 2\delta_\alpha < 0 \quad (79)$$

once the expansion in λ^* is adopted. Note the factor of 2 in this equation caused by QEC.

Whenever Eq. (79) is satisfied, the long-range correlations will produce small corrections to the stochastic error probability. Below, we illustrate this point with an example.

Probability of a “flawless” evolution. Consider the case of a non-Markovian noise model with only one type of error (phase flips, for instance). For simplicity, assume that no spatial correlations exist ($D=0$). Hence, we can consider each qubit separately and do not have to worry about the spatial structure of the quantum computer. We also assume a two-point correlation function of the form

$$\langle f(\mathbf{x}, t_1) f(\mathbf{y}, t_2) \rangle = \frac{1}{2} \left(\frac{\tau_0}{|t_1 - t_2|} \right)^{2\delta/z} \delta_{\mathbf{x}, \mathbf{y}}, \quad (80)$$

where τ_0 is a constant with the dimension of time. How do these long-range correlations change the probability of a flawless evolution of a qubit after $N \gg 1$ QEC steps? To answer this question, we evaluate

$$\begin{aligned} \mathcal{P}(Y_0) &\approx \langle \varphi_0 | \prod_{j=0}^{N-1} v_0^2(\mathbf{x}_j, j\Delta) | \varphi_0 \rangle \\ &\approx (1 - \epsilon)^N \langle \varphi_0 | \prod_{j=0}^{N-1} F_0(\mathbf{x}_j, j\Delta) | \varphi_0 \rangle. \end{aligned} \quad (81)$$

Assuming $\epsilon, \lambda^* \ll 1$, we can rewrite the probability as

$$\begin{aligned} \mathcal{P}(Y_0) &\approx e^{-N\epsilon} \langle \varphi_0 | T_t \exp \left\{ - \frac{[\lambda^* \Delta / (2\hbar)]^2}{1 - \epsilon} \int_0^{N\Delta} \frac{dt}{\Delta} : |f(t)|^2 : \right\} | \varphi_0 \rangle \\ &\approx e^{-N\epsilon} \left\{ 1 + \frac{[\lambda^* \Delta / (2\hbar)]^4}{(1 - \epsilon)^2} \int_0^{N\Delta} \frac{dt_1}{\Delta} \int_0^{t_1} \frac{dt_2}{\Delta} \frac{\tau_0^{4\delta/z}}{(t_1 - t_2)^{4\delta/z}} + \dots \right\} \\ &\approx e^{-N\epsilon} \left\{ 1 + \frac{[\lambda^* \Delta / (2\hbar)]^4}{(1 - \epsilon)^2} \frac{(\tau_0/\Delta)^{4\delta/z} N^{2(1-2\delta/z)}}{2(1-2\delta/z)(1-4\delta/z)} + \dots \right\}, \end{aligned} \quad (82)$$

where we have kept only the leading term. There are two simple limits:

(i) If $z < 2\delta$, the corrections become increasingly irrelevant as N grows. The stochastic probability in the limit of large N is given by $\mathcal{P}(Y_0) \approx e^{-N\epsilon}$ and the correction due to correlations are small.

(ii) The tipping point is $z = 2\delta$. By summing the subset of dominant terms

$$\int_0^{N\Delta} \frac{dt_1}{\Delta} \dots \int_0^{t_{2j}} \frac{dt_{2j+1}}{\Delta} \prod_{i=1}^j \langle : |f(t_{2i-1})|^2 : : |f(t_{2i})|^2 : \rangle, \quad (83)$$

we obtain

$$\mathcal{P}(Y_0) \approx e^{-N\epsilon} \frac{1}{1 - \frac{(\lambda^* \Delta / \hbar)^4}{(1 - \epsilon)^2} \ln N}. \quad (84)$$

This signals a problem with the perturbative expansion when $N \approx \exp(\hbar \frac{1-\epsilon}{\lambda^* \Delta})^2$. For times larger than $\Delta \exp(\hbar \frac{1-\epsilon}{\lambda^* \Delta})^2$, correlations substantially change the probability.

C. Residual decoherence

In addition to the probability of a given syndrome history, we also identified the residual decoherence, Eq. (55), as a fundamental quantity in QEC. The reason is that the noise models that we consider do not satisfy the Laflamme-Knill condition for perfect error correction [12], as is the case for most physically relevant decoherence mechanisms. Hence, it may not be safe to ignore these high-order events in the coupling λ .

It is straightforward to develop a calculation for the density matrix along the same lines used for the syndrome history probability. After separating the intra-hypercube and inter-hypercube contributions, the perturbative expansion is reorganized using the renormalized coupling λ^* . The result is exactly the same as for the case of the probability: If $D+z-2\delta < 0$, the perturbation theory in λ^* is stable and the analysis of the residual decoherence done with the corresponding stochastic model is a good approximation of the true quantum result. We revisit the example used in Sec. III B to make this point clear.

Decoherence of a “flawless” evolution. For this example, we assume an environment that can only introduce phase flip errors in the computer. As we discussed in Sec. I, for this error model we can use the simple three-qubit code. However, unlike the calculation of the probability of a flawless evolution, we now make some assumptions about the spatial structure of the computer: We consider for simplicity that each logical qubit is composed of three adjacent physical qubits. The encoding and decoding are described in Fig. 1.

Following Ref. [39], we write the evolution operator for a particular logical qubit in a QEC cycle as

$$w_0(0, \bar{x}_0) = v_0(\mathbf{x}_1, 0)v_0(\mathbf{x}_2, 0)v_0(\mathbf{x}_3, 0). \quad (85)$$

By expanding Eq. (85) in powers of λ , we obtain

$$\begin{aligned} w_0(0, \bar{x}_0) = & 1 - \left(\frac{\lambda}{2\hbar}\right)^2 \sum_j \int_0^\Delta dt_1 \int_0^{t_1} dt_2 f(\mathbf{x}_j, t_1) f(\mathbf{x}_j, t_2) \\ & + i \left(\frac{\lambda}{2\hbar}\right)^3 \int_0^\Delta dt_1 \int_0^\Delta dt_2 \int_0^\Delta dt_3 f(\mathbf{x}_1, t_1) \\ & \times f(\mathbf{x}_2, t_1) f(\mathbf{x}_3, t_1) \bar{Z}, \end{aligned} \quad (86)$$

where \bar{Z} is the logical phase flip for that particular logical qubit. Note that the third-order term keeps the logical qubit inside the logical Hilbert space [39] and therefore is not corrected by the QEC code.

We choose to evaluate the most off-diagonal term of the reduced density matrix,

$$\rho_{\vec{\uparrow}, \vec{\downarrow}}(Y_0) = \frac{\langle \varphi_0 | [\langle \psi_0 | \prod_{j=N-1}^0 \prod_{k=1}^M w_0^\dagger(j\Delta, \bar{x}_k) | \vec{\downarrow} \rangle \langle \vec{\uparrow} | \prod_{j=0}^{N-1} \prod_{k=1}^M w_0(j\Delta, \bar{x}_k) | \psi_0 \rangle] | \varphi_0 \rangle}{\langle \varphi_0 | \langle \psi_0 | \prod_{j=0}^{N-1} \prod_{k=1}^M w_0^2(\bar{x}_k, j\Delta) | \psi_0 \rangle | \varphi_0 \rangle}, \quad (87)$$

where $\vec{\uparrow} = |\uparrow \cdots \uparrow\rangle$ and $\vec{\downarrow} = |\downarrow \cdots \downarrow\rangle$ denote the state of the physical qubits, \bar{x}_k is labeling M logical qubits, and N is the total number of QEC steps.

After integrating all the modes inside a hypercube, we define a renormalized coupling λ^* and a local error probability ϵ . Finally, we evoke again Wick’s theorem to write

$$\rho_{\vec{\uparrow}, \vec{\downarrow}}(Y_0) = \langle \psi_0 | \vec{\downarrow} \rangle \langle \vec{\uparrow} | \psi_0 \rangle \frac{1 - A - NM\epsilon^3 - \epsilon^4 \left(\frac{\lambda^* \Delta}{2\hbar}\right)^4 \sum_{\mathbf{x}, \mathbf{y}} \int_0^{N\Delta} dt_1 \int_0^{t_1} dt_2 \langle :f^2(\bar{\mathbf{x}}, t_1) :: f^2(\bar{\mathbf{y}}, t_2) : \rangle + \cdots}{1 - A + NM\epsilon^3 + \epsilon^4 \left(\frac{\lambda^* \Delta}{2\hbar}\right)^4 \sum_{\mathbf{x}, \mathbf{y}} \int_0^{N\Delta} dt_1 \int_0^{t_1} dt_2 \langle :f^2(\bar{\mathbf{x}}, t_1) :: f^2(\bar{\mathbf{y}}, t_2) : \rangle + \cdots}, \quad (88)$$

where A is a number proportional to ϵ and λ^* . Hence, for $\epsilon, \lambda^* \ll 1$, this simplifies to [53]

$$\rho_{\vec{\uparrow}, \vec{\downarrow}}(Y_0) \approx \langle \psi_0 | \vec{\downarrow} \rangle \langle \vec{\uparrow} | \psi_0 \rangle \left[1 - 2NM\epsilon^3 - 2\epsilon^4 \left(\frac{\lambda^* \Delta}{2\hbar}\right)^4 \int d\mathbf{x} \int d\mathbf{y} \int_0^{N\Delta} dt_1 \int_0^{t_1} dt_2 \langle :f^2(\bar{\mathbf{x}}, t_1) :: f^2(\bar{\mathbf{y}}, t_2) : \rangle + \cdots \right]. \quad (89)$$

If we now recall the two-point correlation function of Eq. (7), it becomes clear that the corrections due to correlations are relevant when $D+z > 2\delta$.

D. Relation to the work of Aharonov, Kitaev, and Preskill

The study of correlated noise has been a central problem for quite some time. Among the most recent advances is a paper by Aharonov, Kitaev, and Preskill (AKP) [26]. Using a method completely different from ours, AKP proved the following: For a computer where qubits are interacting through an instantaneous interaction of the form $\lambda^2/\Delta x^{2\delta}$, it is possible to prove resilience for $\lambda < \lambda_c$ and $D-2\delta < 0$. The key distinction between the work of AKP and ours is the instantaneous nature of their interaction. Hence, while in our work each qubit is inside a distinct hypercube, for AKP they are all

contained in a single hypercube. There is however a trade-off. Since their interaction is instantaneous and perfect error correction is assumed, there is no propagation of errors in time through the gauge field of the environment. Hence, effectively, AKP are considering a model with $z=0$. As a result, our Eq. (79) holds in the case they analyzed as well.

V. THRESHOLD THEOREM AS A QUANTUM PHASE TRANSITION

The main result of fault-tolerant quantum computation is the threshold theorem. The theorem states that if a stochastic error probability ϵ is smaller than a critical value ϵ_c , then the introduction of an additional layer of concatenation improves the protection of the information. Hence, for a fixed ϵ , it is possible to sustain a quantum computation for any desire

time at the cost of some reasonable additional hardware overhead.

Even though quantum computation and QEC are out-of-equilibrium problems, it is intuitive to talk about different phases in the computer-environment parameter space. Along this line of thought, each phase corresponds to a distinct steady state. A natural choice for an order parameter is that given by the entanglement among the qubits and the environment. We summarize our thinking in Fig. 5, where we present a schematic phase diagram for a quantum computer running QEC.

For stochastic noise models, such an idea was explored by Aharonov [18]. Following that work, we can separate the behavior of the computer into two distinct regimes:

(i) For $\epsilon < \epsilon_c$, the computer components can maintain large entanglement through fault-tolerant procedures, which in turns means that the computer and the environment are weakly entangled. Hence, due to this large internal entanglement, the quantum computer departs from the classical computer model. We can formalize these remarks by remembering that QEC tries to keep the system in the “steady state” described by the reduced density matrix. In order to keep the notation simple, let us take the ideal computer state as a pure state,

$$\rho(t) = |\psi\rangle\langle\psi|, \quad (90)$$

with $|\psi\rangle = \sum_i \alpha_i(t) |i\rangle$ expressed in terms of the computational basis $\{|i\rangle\}$. As a consequence, it has a reduced entropy $S \approx 0$. In this case, if we look at the full Hilbert space (that is, before tracing out the environment), we find the tensor state

$$|\Psi\rangle \approx |\psi\rangle \otimes |\varphi_{\text{environment}}\rangle. \quad (91)$$

(ii) For $\epsilon > \epsilon_c$, the computer components are weakly entangled and, therefore, can be efficiently simulated by a Turing machine. In other words, the computer density matrix no longer represents a pure state, but rather a statistical mixture. Thus, the computer components are strongly entangled with the environment. This corresponds to a steady state with a large reduced entropy (in the limit of $\epsilon \rightarrow 1$, $S \approx N \ln 2$, with N the number of qubits).

In such a description, we see that ϵ plays a role analogous to an effective temperature [54]. Hence, the threshold theorem defines a phase transition from a high-temperature phase, where qubits are independent from each other, to a low-temperature phase, where quantum coherence and entanglement are possible [18]. This also sheds new light on the role of periodic measurements in QEC: They can be seen as a refrigeration that extracts entropy from the computer (very much like the Schulman-Vazirani initialization procedure [55] or the transfer of entanglement to fresh ancillas [34]). If the entropy production in the computer is below a certain level, then the computer can be kept in its “low-temperature” phase.

Our analysis of correlated noise also fits perfectly into this description. The dimension criterion provided by Eq. (79) is the hallmark of a quantum phase transition [56]. For $D+z < 2\delta$, V can only produce small corrections to the stochastic error model. The steady state of the system is therefore given by Eq. (91). There is a clear separation of scales and the

threshold theorem holds as it is. Conversely, for $D+z > 2\delta$, there is no clear separation of scales. The computer and the environment become increasingly entangled and the system is driven toward a different steady state. Such a state is probably distinct from the “high temperature” one and it is likely that it is characterized by a smaller residual entropy.

This does not mean that for $D+z > 2\delta$ it would not be possible to perform quantum computation. It only means that the threshold theorem as we stated it does not hold. It is conceivable that some different derivation of the theorem exists in this case. In this sense, $D+z=2\delta$ defines what is usually referred to as the upper critical dimension of the model (see Appendix B). Below the upper critical dimension, there can be substantial corrections to the steady state given by Eq. (91), but it may still be possible to prove resilience. The question that remains open is whether a lower critical dimension exists, namely, a criterion for V that would tell us when it is impossible to perform long-time quantum computation.

VI. SUMMARY AND CONCLUSIONS

Most previous discussions of QEC have used the quantum master equation and quantum dynamical semigroups [57]. This is a very natural approach: The computer is the object of interest; hence, one starts the discussion by integrating out the environmental degrees of freedom. However, the price paid in this approach is that some simplification is needed in order to derive the quantum master equation [57,25]. The usual assumption is the Born-Markov approximation [25]. In that case, it is natural to define an error probability for a given qubit, and a discussion in terms of error models naturally follows [1,29]. The situation is much less clear when the Born-Markov approximation cannot be justified [28,58]. In this case, temporal and spatial correlations can build up and completely destroy the notion of the probability of an error.

A key characteristic of the discussion here is that we do not try to use a quantum master equation. Rather, we follow the approach put forward by Schwinger and Keldysh [42–44]

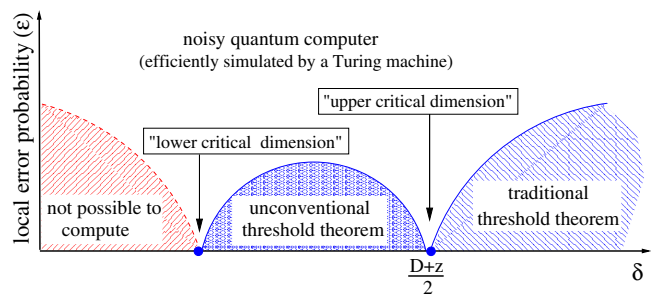


FIG. 5. (Color online) Phase diagram of a quantum computer running QEC. The parameter δ is the scaling dimension of the environment operator, D is the dimensionality of the computer, and z is the dynamical exponent of the environment [see discussion preceding Eq. (79)]. In the red phase, qubits and environment are strongly entangled causing strong decoherence. In the light blue phase, QEC keeps the qubits and environment disentangled, making computation possible.

to study out of equilibrium systems. The main conceptual difference is that we trace the environmental degrees of freedom only at the very last step of the calculation. Hence, we can make the most of the unitary evolution of a quantum mechanical system.

Following this ‘‘Schwinger-Keldysh’’ approach, we discussed the evolution of a quantum computer operated with fast and slow gates. On the one hand, for fast gates the microscopic Hamiltonian is the one relevant for the evolution of the computer, Eq. (23). On the other hand, for slow gates we demonstrated that a suitable effective Hamiltonian, Eq. (40), can be used to provide an upper bound for the discussion of decoherence. With this effective Hamiltonian, the notation can be unified, and both cases treated simultaneously. We derived two formal expressions that quantify the evolution of the computer under QEC in a correlated environment: (i) The probability of a given syndrome history, Eq. (54), and (ii) the reduced density matrix of the computer, Eq. (55).

In order to fully use standard QEC theory, we introduced the important assumption of ‘‘hypercubes,’’ that is a minimum spatial distance between qubits, Eq. (57), in order to allow the definition of an error probability for a single qubit. With this ‘‘hypercube assumption,’’ it is straightforward to use Wick’s theorem to separate the environmental modes into intrahypercube and interhypercube parts. The intrahypercube component defines the error probability, while the interhypercube part is tracked by an operator acting on the coarse-grained scale of the hypercubes. As examples, we treated a generalization of the spin-boson model and a quantum frustrated model.

All the pieces are put together when we explicitly calculate the probability of a syndrome history (Sec. IV B) and associated residual decoherence (Sec. IV C). The main result is cast as a dimensional criterion, Eq. (79). Finally, we discuss the parallels between the threshold theorem and a quantum phase transition. A qualitative description of the possible fates of a quantum computer as a function of noise strength and degree of correlation is given in Fig. 5.

There are several clear directions in which our results could be extended or improved. First, it would obviously be desirable to relax the hypercube assumption introduced in Sec. IV. There is nothing intrinsic to our approach which makes this assumption necessary. Yet, progress without it seems much more difficult: The notion of a local error probability during a single QEC cycle becomes problematic, making the connection with analysis based on error models, such as the usual derivation of the threshold theorem, unclear.

Second, noninstantaneous gate operation is clearly a delicate issue. By using a bound (Sec. II C), we are able to treat this case in the same way as the fast-gate case. Thus we derive an upper bound for the local error probability together with the dimensional criterion. If a more accurate value for the error probability is desired, a specific error correction code as well as the gates under consideration must be included in the analysis. However, the scaling argument and resulting dimensional criterion do not, in general, change.

Note that it is possible to change the dimensional criterion for the better (but not for the worse) by using the separation of scales introduced by QEC. Particular pulse sequences can

reduce correlation at long times at the cost of increasing the local error probability. One example was given in our previous work [39,40].

Finally, there may be a regime of parameters where, as indicated in Fig. 5, fault-tolerant quantum computation is possible even though the presently known derivations of the threshold theorem do not apply. By analogy with phase transition phenomenology, there may be a lower critical dimension such that a more sophisticated analysis than the one we present here shows that fault-tolerant computation is possible for $\delta < (D+z)/2$. It would be very interesting to show in any example that such is, or is not, the case.

Quantum error correction is one of the most interesting frameworks which allows long quantum computations [59]. Even though QEC is widely accepted, it has been argued that it relies on a set of unphysical assumptions [22,25,60,61], namely, (i) ‘‘fast’’ measurements, (ii) ‘‘fast’’ gates, and (iii) describing decoherence by error models. Although these are legitimate concerns, it is now clear that they are not fundamental: First, in Ref. [35], DiVincenzo and Aliferis demonstrated that resilient circuits can be constructed with slow measurements. Second, in the present paper, we have demonstrated that the fast-gate assumption is not critical for fault tolerance. Finally, we have laid the groundwork here for a theoretical framework that connects microscopic Hamiltonians with error models in correlated environments. From our results for the threshold theorem in conjunction with those of AKP [26], it is clear that a large class of correlated environments are already properly treated within the QEC framework.

ACKNOWLEDGMENTS

We thank C. Kane, D. Khveshchenko, and R. Plesser for useful discussions. This work was supported in part by NSF Grants No. CCF 0523509 and No. CCF 0523603. E.R.M. acknowledges partial support from the Interdisciplinary Information Science and Technology Laboratory at UCF.

APPENDIX A: ABSOLUTE CONVERGENCE OF THE DYSON SERIES

The Dyson series is absolutely convergent for any bound operator evolving for any finite time [62]. This is particularly simple to see using the ‘‘sup’’ operator norm [28],

$$\|A\| = \sup_{\Psi} \sqrt{\langle \Psi | A^\dagger A | \Psi \rangle}, \quad (\text{A1})$$

where $\|\Psi\| = 1$. If $P = \int_0^t dt' \|V(t')\| < \infty$, then the norm of the m th-order term in Dyson’s series is bounded by $P^m/m!$. Thus, using the convergence of the exponential series, we find that Dyson’s series is absolutely convergent.

APPENDIX B: PERTURBATIVE EXPANSION IN ϕ^4 THEORY

A classic example of a quantum phase transition is given by the ϕ^4 theory at criticality [63]. The model is compactly described by the Euclidean action

$$S = \int_0^L d^D r \int_0^\beta d\tau [(\nabla_r \phi)^2 + (\partial_\tau \phi)^2 + \lambda \phi^4]. \quad (\text{B1})$$

The scaling dimension of the free field is usually defined as $\dim[\phi] = \nu/2$. If we expand the partition function in powers of λ , it is simple to see that each order in the perturbative expansion will have the power $\lambda(L\beta)^{D+1-\nu}$. Hence, $D+1-\nu < 0$ is the criterion for the irrelevance of the perturbation. The simplest way to see that is to do power counting by rescaling space and time,

$$r \rightarrow br, \quad \tau \rightarrow b\tau, \quad \phi \rightarrow b^{-\nu/2} \phi, \quad (\text{B2})$$

which immediately gives

$$S = b^{D-1-\nu} \int d^D r \int d\tau [(\nabla_r \phi)^2 + (\partial_\tau \phi)^2] + \lambda b^{D+1-2\nu} \int d^D r \int d\tau \phi^4. \quad (\text{B3})$$

One finds the scaling $\lambda \rightarrow \lambda b^{D+1-\nu}$, which is valid at each order of the perturbative expansion. The criterion for the irrelevance of the perturbation is $D+1-\nu < 0$.

There is one more important definition that this example provides. Since the Gaussian action must be scale invariant, we automatically see that for this example $\nu = D-1$. Hence, the criteria for the irrelevance of $\lambda \phi^4$ term as a perturbation can be rewritten as $3-D < 0$. This defines the upper critical dimension for the model as $d_c^{\text{upper}} = 4$ (three spatial and one temporal). When a system is above its upper critical dimension, the physics is controlled by the Gaussian action. However, when the system is below its upper critical dimension, there are substantial corrections to physical quantities when compared with the Gaussian solution.

APPENDIX C: HILBERT SPACE OF QUBITS

Due to the state vector normalization, the Hilbert space of a qubit is isomorphic to a three-dimensional sphere S^3 : For a general state $|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$, we have the constraint

$$(\text{Re } \alpha)^2 + (\text{Im } \alpha)^2 + (\text{Re } \beta)^2 + (\text{Im } \beta)^2 = 1. \quad (\text{C1})$$

However, an overall phase is physically irrelevant and the correct mapping is to the complex projective plane of complex dimension 1,

$$S^3/\text{U}(1) \rightarrow \mathbb{C}\text{P}^1. \quad (\text{C2})$$

For the same reason, the Hilbert space of n qubits is isomorphic to $\mathbb{C}\text{P}^{2n-1}$. For the discussion of entanglement, there is a particularly important subspace of this space. It is composed by the direct product of each qubit Hilbert space minus an overall phase,

$$\prod_{j=1}^n \mathbb{C}\text{P}^1_{(j)} \Big|_{\text{modulus phase}} \subset \mathbb{C}\text{P}^{2n-1}, \quad (\text{C3})$$

where j labels the j th qubit's Hilbert space. The dimension of the subspace grows as $n-1$ while the dimension of the entire

Hilbert space grows as $2n-1$. Entangled states are defined as the complementary set of this special subspace.

APPENDIX D: DECOHERENCE IN THE SPIN-BOSON MODEL WITH OHMIC DISSIPATION

An example of a qubit coupled to an environment is the spin-boson model with ohmic dissipation [64,65], which was intensively studied in the context of quantum computation [2,66] even before quantum error correction was introduced. In this model, a qubit evolves according to the Hamiltonian

$$H = \int dx [(\partial_x \phi)^2 + \Pi^2] + \lambda \partial_x \phi(0) \sigma^z, \quad (\text{D1})$$

where ϕ is a chiral bosonic field, $\vec{\sigma}$ are Pauli matrices that describe the qubit located at $x=0$, and λ is the environment-qubit coupling constant. If a qubit is prepared in an initial state

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle, \quad (\text{D2})$$

at large enough times, $\Lambda^{-1} \ll t \ll (k_B T)^{-1}$, its density matrix evolves as

$$\rho(t) = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* e^{-\lambda^2 \ln(1+\Lambda t)} \\ \alpha^* \beta e^{-\lambda^2 \ln(1+\Lambda t)} & |\beta|^2 \end{bmatrix}, \quad (\text{D3})$$

with Λ denoting the environment ultraviolet cutoff frequency. Since states with either α or β equal to 0 do not experience decoherence, they are called classical states. They define a pointer basis. Conversely, any superposition state with $\alpha, \beta \neq 0$ suffers decoherence and over a long time becomes a statistical mixture of the classical states.

As one includes more qubits, the entries in the reduced density matrix will decay faster as one moves away from the diagonal. In the case where qubits are coupled to independent baths, it is simple to see that the off-diagonal matrix elements decay as

$$\rho_{\vec{p}, \vec{q}}(t \gg \Lambda^{-1}) = \rho_0 e^{-\lambda^2 (p-q) \ln(1+\Lambda t)}, \quad (\text{D4})$$

where p and q are the total magnetization of the states \vec{p} and \vec{q} , respectively [2]. The case of a common bath is also straightforward [2], and the result for qubits separated by a distance smaller than Λ^{-1} is

$$\rho_{\vec{p}, \vec{q}}(t \gg \Lambda^{-1}) \approx \rho_0 e^{-\lambda^2 (p-q)^2 \ln(1+\Lambda t)}. \quad (\text{D5})$$

Some entangled states do not suffer decoherence (a singlet state, for example). However, these correspond to a very special and small decoherence-free subspace. In general, entangled states are made of quantum superpositions and therefore have components in the off-diagonal entries of the density matrix. Hence, studying decoherence (the decay of the off-diagonal elements of the density matrix) is essentially equivalent to studying how entanglement between qubits is destroyed by interaction with the environment.

APPENDIX E: INTERACTION PICTURE

Since $[H_0, H_{\text{QC}}] = 0$, we can define the interaction picture

$$O(t) = e^{iH_0 t/\hbar} R^\dagger(t) O R(t) e^{-iH_0 t/\hbar}, \quad (\text{E1})$$

$$|\Psi(t)\rangle = e^{iH_0 t/\hbar} R^\dagger(t) \tilde{U}(t) |\Psi(0)\rangle, \quad (\text{E2})$$

where $\tilde{U}(t)$ is the exact evolution operator, defined as

$$U(t) = T_t e^{-i\hbar \int_0^t dt' H(t')}, \quad (\text{E3})$$

and $|\Psi\rangle$ is the total state vector (computer plus environment). Now, let us consider the time evolution of $|\Psi\rangle$,

$$\frac{d}{dt} |\Psi(t)\rangle = \frac{d}{dt} e^{iH_0 t/\hbar} R^\dagger(t) \tilde{U}(t) |\Psi(0)\rangle = -\frac{i}{\hbar} V(t) |\Psi(t)\rangle. \quad (\text{E4})$$

Thus, we obtain the usual definition for the evolution operator in the interaction picture

$$\tilde{U}(t) = e^{iH_0 t/\hbar} R^\dagger(t) \tilde{U}(t) = T_t e^{-i\hbar \int_0^t dt' V(t')}. \quad (\text{E5})$$

APPENDIX F: LOW-FREQUENCY CONTRIBUTION TO THE ERROR PROBABILITY

The simplest way to understand F_α is to write f in its frequency representation

$$\begin{aligned} v_\alpha(\mathbf{x}_1, \lambda_\alpha^*) &\approx \lambda_\alpha^* \int_0^\Delta dt f_\alpha(\mathbf{x}_1, t) \approx \lambda_\alpha^* \int_0^\Delta dt \int_0^\Lambda d\omega e^{i\omega t} f_\alpha(\mathbf{x}_1, \omega) \\ &\approx \lambda_\alpha^* \int_0^\Delta dt \left(\int_0^{\Delta^{-1}} d\omega + \int_{\Delta^{-1}}^\Lambda d\omega \right) e^{i\omega t} f_\alpha(\mathbf{x}_1, \omega) \\ &\approx \lambda_\alpha^* \int_0^\Delta dt [f_\alpha^>(\mathbf{x}_1, t) + f_\alpha^<(\mathbf{x}_1, 0)], \end{aligned} \quad (\text{F1})$$

where the less than symbol stands for frequencies smaller than Δ^{-1} and the greater than symbol stands for the frequencies between Δ^{-1} and Λ . Thus, using that $\langle f_\alpha^< f_\alpha^> \rangle = 0$, we obtain

$$\begin{aligned} v_\alpha^2(\mathbf{x}_1, \lambda_\alpha^*) &\approx (\lambda_\alpha^*)^2 \int_0^\Delta dt_1 dt_2 f_\alpha^>^\dagger(\mathbf{x}_1, t_1) f_\alpha^>(\mathbf{x}_1, t_2) \\ &\quad + (\lambda_\alpha^* \Delta)^2 f_\alpha^<^\dagger(\mathbf{x}_1, 0) f_\alpha^<(\mathbf{x}_1, 0). \end{aligned} \quad (\text{F2})$$

APPENDIX G: SCALING DIMENSION OF F_α

If the two-point correlation function of f_α can be expressed as

$$\langle f_\alpha(\mathbf{x}_1, t_1) f_\alpha(\mathbf{x}_2, t_2) \rangle \sim \mathcal{F} \left(\frac{1}{(\Delta x)^{2\delta}}, \frac{1}{(\Delta t)^{2\delta z}} \right), \quad (\text{G1})$$

the scaling dimension of f_α is defined as $\dim f_\alpha = \delta$. Using Wick's theorem,

$$\begin{aligned} \langle :|f_\alpha(\mathbf{x}_1, t_1)|^2 : |f_\alpha(\mathbf{x}_2, t_2)|^2 : \rangle &= \langle f_\alpha^\dagger(\mathbf{x}_1, t_1) f_\alpha^\dagger(\mathbf{x}_2, t_2) \rangle \langle f_\alpha(\mathbf{x}_1, t_1) f_\alpha(\mathbf{x}_2, t_2) \rangle \\ &\quad + \langle f_\alpha^\dagger(\mathbf{x}_1, t_1) f_\alpha(\mathbf{x}_2, t_2) \rangle \langle f_\alpha(\mathbf{x}_1, t_1) f_\alpha^\dagger(\mathbf{x}_2, t_2) \rangle \\ &= 2 \left[\mathcal{F} \left(\frac{1}{(\Delta x)^{2\delta}}, \frac{1}{(\Delta t)^{2\delta z}} \right) \right]^2. \end{aligned} \quad (\text{G2})$$

Therefore, $\dim F_\alpha = 2\delta$.

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