

## Resilient Quantum Computation in Correlated Environments: A Quantum Phase Transition Perspective

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We analyze the problem of a quantum computer in a correlated environment protected from decoherence by quantum error correction using a perturbative renormalization group approach. The scaling equation obtained reflects the competition between the dimension of the computer and the scaling dimension of the correlations. For an irrelevant flow, the error probability is reduced to a stochastic form for a long time and/or a large number of qubits; thus, the traditional derivation of the threshold theorem holds for these error models. In this way, the “threshold theorem” of quantum computing is rephrased as a dimensional criterion.

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A central result in the theory of quantum error correction (QEC) is the “threshold theorem” [1]. Even though QEC is a perturbative method [1,2], the threshold theorem states that, provided the noise strength is below a critical value, quantum information can be protected for arbitrarily long times. This remarkable theorem was first derived for stochastic error models [3] or, more generally, for environments where the spatial and temporal correlations have an exponential decay [4]. Although there is still some controversy [5], the theorem is widely accepted by a community where the foremost effort has been to extend it to correlated error models [6–9].

The key difference between a correlated error model and a stochastic one is the absence of a local and time independent error probability. The main concept in extending the theorem to a correlated environment has been the norm of an operator [6–8]. A convenient norm of the interaction Hamiltonian can be used to bound the error probability and, eventually, to prove the threshold theorem. However, for some models the norm of the interacting Hamiltonian is too large to provide useful bounds [6]. In this Letter, we hence choose to follow different reasoning.

We demonstrate that a large class of correlated error models is reduced to a simple stochastic error model in the asymptotic limit of a large number of qubits or a long time. Thus, in order to prove the resilience of the quantum information, we can fall back on the traditional derivation of the threshold theorem. Because the conditions for this fallback have clear parallels with the theory of quantum phase transitions [10,11], we rephrase the threshold theorem as a dimensional criterion: (i) For systems above their “critical dimension,” the traditional proof of resilience is valid, and there are two regimes, or phases, as a function of the coupling with the environment; (ii) however, when the system is below its critical dimension, the effects of correlations produce large corrections, and it is not possible to prove resilience by our arguments.

In an elegant Letter, Aharonov, Kitaev, and Preskill (AKP) [8] derived a new set of “threshold conditions” for a correlated noise model with instantaneous power law interactions between any two qubits of the computer. AKP proved that a version of the theorem holds if (i) the coupling is below a critical value, and (ii) the interaction between the qubits decays sufficiently fast when compared with the number of spatial dimensions of the computer. The first condition is very much what one expects for the threshold theorem. However, the second is very suggestive of our interpretation of the threshold theorem in terms of a quantum phase transition. Even though AKP considered an error model substantially different from ours (see below), a particular case of our discussion reproduces AKP’s conditions for resilience.

*The error model.*—We consider an environment described by a “noninteracting” field theory  $H_0$ , allowing application of Wick’s theorem. Furthermore, since an environment with a spectral gap would imply exponentially decaying correlations [4], we assume that the environment is gapless and has wave velocity  $v$  and short time cutoff  $\Lambda^{-1}$ . We focus on an interaction Hamiltonian that is local in the qubits

$$V = \sum_{\mathbf{x}, \alpha} \lambda_{\alpha} f_{\alpha}(\mathbf{x}) \sigma_{\alpha}(\mathbf{x}), \quad (1)$$

where  $\vec{f}$  is a function of the environment degrees of freedom and  $\vec{\sigma}$  are the Pauli matrices that parametrize the qubits. Finally, we consider (without loss of generality) the qubits to be arranged in a  $D$ -dimensional lattice.

The evolution of the system in the interaction picture during a QEC cycle is given by

$$\hat{U}(\Delta, \lambda_{\alpha}) = T_t e^{-i \int_0^{\Delta} dt \sum_{\mathbf{x}, \alpha} \lambda_{\alpha} f_{\alpha}(\mathbf{x}, t) \sigma_{\alpha}(\mathbf{x})}, \quad (2)$$

with  $\Delta$  corresponding to the time of the syndrome extraction,  $T_t$  the time ordering operator, and  $f_{\alpha}(\mathbf{x}, t) = e^{(i/\hbar)H_0 t} f_{\alpha}(\mathbf{x}) e^{-(i/\hbar)H_0 t}$ . At time  $\Delta$ , the extraction of syn-

dromes selects some terms of Eq. (2) as the evolution operator for that particular QEC cycle [12]. Hence, we naturally define a coarse-grained space-time grid of hypercubes  $\Delta \times (\nu\Delta)^{D/z}$  (with  $z$  being the dynamical exponent of the environment).

The coarse grain grid is the frame upon which we develop our analysis. It provides the scale that separates two distinct noise regimes: intra- and interhypercube components. The key simplifying assumption is that in each volume  $(\nu\Delta)^{D/z}$  there is only *one* qubit. In this case, the intrahypercube part is simply the probability  $\epsilon_\alpha$  of an error of type  $\sigma_\alpha$  on that particular qubit. Two physical systems where this hypothesis should be immediately valid are acoustic phonons interacting with solid-state qubits and Ohmic noise due to voltage or current fluctuations on qubits based on quantum dots or superconducting devices.

Our discussion is therefore divided into two parts. First, we demonstrate how to calculate this “stochastic” error probability  $\epsilon_\alpha$ . Second, we evaluate how this error proba-

bility is changed by the interhypercube component of the noise.

*Defining  $\epsilon_\alpha$ .*—Consider, for instance, that we know from the syndrome that a particular qubit, say,  $\mathbf{x}_1$ , suffered an error  $\sigma_\alpha$  in the time interval labeled 0. To lowest order in  $\lambda_\alpha$ , the QEC code disentangles the qubit from the environment. Thus,

$$\hat{v}_\alpha(\mathbf{x}_1, \lambda_\alpha) \approx -i\lambda_\alpha \int_0^\Delta dt f_\alpha(\mathbf{x}_1, t) \quad (3)$$

is the evolution operator associated with that particular qubit, where we used that  $\sigma_\alpha(\Delta)\sigma_\alpha(t) = \mathbf{1}$  [13].

Of course, lowest-order perturbation theory is not always justified. Hence, before QEC can be argued to be effective, it is important to estimate how higher-order terms change Eq. (3). A very direct approach is to use the perturbative renormalization group (RG).

To derive the RG equations, we supplement (3) with the next higher-order terms allowed by the QEC code

$$\begin{aligned} \hat{v}_\alpha(\mathbf{x}_1, \lambda_\alpha) \approx & -i\lambda_\alpha \int_0^\Delta dt f_\alpha(\mathbf{x}_1, t) - \frac{1}{2} |\epsilon_{\alpha\beta\gamma}| \lambda_\beta \lambda_\gamma \sigma_\alpha(\Delta) T_t \int_0^\Delta dt_1 dt_2 f_\beta(\mathbf{x}_1, t_1) f_\gamma(\mathbf{x}_1, t_2) \sigma_\beta(t_1) \sigma_\gamma(t_2) \\ & + \frac{i}{6} \sum_\beta \lambda_\alpha \lambda_\beta^2 \sigma_\alpha(\Delta) T_t \int_0^\Delta dt_1 dt_2 dt_3 f_\alpha(\mathbf{x}_1, t_1) f_\beta(\mathbf{x}_1, t_2) f_\beta(\mathbf{x}_1, t_3) \sigma_\alpha(t_1) \sigma_\beta(t_2) \sigma_\beta(t_3), \end{aligned} \quad (4)$$

where  $\epsilon_{\alpha\beta\gamma}$  is the antisymmetric tensor. Since QEC completely removes the qubit variable from the problem, we can consider Eq. (4) as a field theory problem in itself. It is therefore straightforward to derive the lowest-order terms to the  $\beta$  function of  $\hat{v}_\alpha(\mathbf{x}_1)$ :

$$\frac{d\lambda_\alpha}{d\ell} = g_{\beta\gamma}(\ell) \lambda_\beta \lambda_\gamma + \sum_\beta h_{\alpha\beta}(\ell) \lambda_\alpha \lambda_\beta^2, \quad (5)$$

where  $g$  and  $h$  are functions specific to a particular environment and  $d\ell = d\Lambda/\Lambda$  [14]. Note that the integration of Eq. (5) is equivalent to summing an infinite series of terms in the perturbative expansion for  $\hat{v}_\alpha(\mathbf{x}_1)$ . Because there is only one qubit in the hypercube, the problem of calculating  $\epsilon_\alpha$  involves only  $f(\mathbf{x}_1)$  and so was reduced to an impurity problem.

To further proceed with the argument, we must ask that the renormalized value of  $\lambda_\alpha$  at the  $(\nu\Delta)^{-1}$  scale  $\lambda_\alpha^*$  be a small parameter. In that case, it is appropriate to use the evolution equation (3) with  $\lambda_\alpha$  replaced by  $\lambda_\alpha^*$ .

The conditional probability of having an error of type  $\alpha$  in a particular hypercube labeled by  $\mathbf{x}_1$  and  $t = 0$  has the general form [12]

$$P(\dots; \alpha, \mathbf{x}_1; \dots) \approx \langle \dots \hat{v}_\alpha^\dagger(\mathbf{x}_1, \lambda_\alpha^*) \dots \hat{v}_\alpha(\mathbf{x}_1, \lambda_\alpha^*) \dots \rangle.$$

Hence, following the discussion in Ref. [12], we define the operator that gives the probability of an error as

$$v_\alpha^2(\mathbf{x}_1, \lambda_\alpha^*) \approx \hat{v}_\alpha^\dagger(\mathbf{x}_1, \lambda_\alpha^*) \hat{v}_\alpha(\mathbf{x}_1, \lambda_\alpha^*). \quad (6)$$

We can now readily separate the effects of correlations into

their intra- and interhypercube parts. Using Wick’s theorem, we obtain

$$v_\alpha^2(\mathbf{x}_1, \lambda_\alpha^*) \approx \epsilon_\alpha + (\lambda_\alpha^* \Delta)^2 :|f_\alpha(\mathbf{x}_1, 0)|^2:, \quad (7)$$

where the intrahypercube part is

$$\epsilon_\alpha = (\lambda_\alpha^*)^2 \int_0^\Delta dt_1 \int_0^\Delta dt_2 \langle f_\alpha^\dagger(\mathbf{x}_1, t_1) f_\alpha(\mathbf{x}_1, t_2) \rangle \quad (8)$$

and  $::$  stands for normal ordering. An important remark is that  $\epsilon_\alpha$  is from this point on a numerical factor set by  $\lambda^*$  at the scale  $(\nu\Delta)^{-1}$ .

For later convenience, we rewrite Eq. (7) in a slightly different form. Defining the operators

$$F_\alpha(\mathbf{x}_1, 0) = \frac{(\lambda_\alpha^* \Delta)^2}{\epsilon_\alpha} :|f_\alpha(\mathbf{x}_1, 0)|^2:, \quad (9)$$

we rewrite the operator for the probability of an error as the product  $v_\alpha^2(\mathbf{x}_1, \lambda_\alpha^*) = \epsilon_\alpha [1 + F_\alpha(\mathbf{x}_1, 0)]$ . By direct calculation, or simply by the unitarity of the probability, it is also straightforward to write the operator for the probability of not having an error as  $v_0^2(\mathbf{x}_1, \lambda_\alpha^*) = [1 - \sum_\alpha \epsilon_\alpha] F_0(\mathbf{x}_1, 0)$ , where

$$F_0(\mathbf{x}_1, 0) = 1 - \frac{\sum_\beta (\lambda_\beta^* \Delta)^2 :|f_\beta(\mathbf{x}_1, 0)|^2:}{1 - \sum_{\beta=x,y,z} \epsilon_\beta}. \quad (10)$$

*Probability function and scaling.*—Now that we separated the probability into intra- and interhypercube components, we can seek to write the probability of a computer evolution with  $m$  errors after  $N$  QEC cycles. A particular

case is helpful in understanding how to proceed. Suppose we want to calculate the probability that  $R$  qubits suffer  $m \ll NR$  errors of type  $\alpha$ . Using Eqs. (7)–(9), it is straightforward to write

$$P_m^\alpha = P_m \int \frac{d\mathbf{x}_1}{(v\Delta)^{D/z}} \cdots \frac{d\mathbf{x}_m}{(v\Delta)^{D/z}} \int_0^{N\Delta} \frac{dt_1}{\Delta} \cdots \int_0^{t_{m-1}} \frac{dt_m}{\Delta} \\ \times \left\langle \left[ \prod_{\zeta} F_0(\mathbf{x}_\zeta, t_\zeta) \right] \right. \\ \left. \times [1 + F_\alpha(\mathbf{x}_1, t_1)] \cdots [1 + F_\alpha(\mathbf{x}_m, t_m)] \right\rangle, \quad (11)$$

where we integrated over all possible grid positions  $(\mathbf{x}_j, t_j)$ ,  $\zeta$  denotes the set of remaining hypercubes, and  $p_m = (1 - \sum_\alpha \epsilon_\alpha)^{RN-m} (\epsilon_\alpha)^m$  [15].

We now organize the expectation value of Eq. (11) in powers of  $(\lambda^*, \epsilon_\alpha)$  and invoke Wick's theorem again. The first term is just the stochastic contribution to the probability:

$$P_m \int \prod_{k=1}^m \frac{d\mathbf{x}_k}{(v\Delta)^{D/z}} \frac{dt_k}{\Delta} = P_m \binom{NR}{m} \sim P_m (NR)^m. \quad (12)$$

The next term is typically of the form

$$P_m \int \prod_{k=1}^m \frac{d\mathbf{x}_k}{(v\Delta)^{D/z}} \frac{dt_k}{\Delta} \langle F_\alpha(\mathbf{x}_i, t_i) F_\alpha(\mathbf{x}_j, t_j) \rangle. \quad (13)$$

Thus, the fundamental role of the scaling dimension of  $F_\alpha$  now becomes clear. If  $\dim[f_\alpha] = \delta_\alpha$ , then Eq. (9) implies that  $\dim[F_\alpha] = 2\delta_\alpha$ . Since  $H_0$  is assumed to be noninteracting, the two point correlation function has the general form

$$\langle F_\alpha(\mathbf{x}_i, t_i) F_\alpha(\mathbf{x}_j, t_j) \rangle \sim \mathcal{F}(|\mathbf{x}_i - \mathbf{x}_j|^{-4\delta_\alpha}, |t_i - t_j|^{-4\delta_\alpha/z}).$$

For large  $N$  or  $R$ , we can study the stability of the expansion of Eq. (11) in powers of  $\lambda^*$  using the traditional scaling theory [10]. The simplest way to proceed is to apply the transformation  $\mathbf{x}_{i,j} \rightarrow b\mathbf{x}'_{i,j}$  and  $t_{i,j} \rightarrow b^z t'_{i,j}$ , with  $b = e^{d\ell}$ , to Eq. (13). It is then straightforward to obtain the scaling equation for  $\lambda^*$ :

$$\frac{d\lambda_\alpha^*}{d\ell} = (D + z - \dim[F_\alpha])\lambda_\alpha^*. \quad (14)$$

A similar argument can be applied to all of the remaining terms of Eq. (11). The critical condition is when  $D + z = 2\delta$ . In this particular case, correlations between hypercubes introduce logarithmic corrections to the stochastic part of Eq. (11).

An irrelevant flow for  $\lambda^*$  indicates that the system is above its critical dimension. Since correlations between hypercubes produce small corrections in comparison to Eq. (12), the probability distribution at a long time and/or a large number of qubits has essentially a stochastic form. Corrections to this form can be systematically calculated by perturbative expansion in  $\lambda_\alpha^*$ . Thus, we can fall

back on the usual proof of resilience for quantum information [3,4]: There will be two ‘‘phases’’ separated by a  $\lambda_\alpha^{\text{critical}}$ : (i) For  $\lambda_\alpha < \lambda_\alpha^{\text{critical}}$ , the information can be protected for arbitrarily long times. Hence, the computer and the environment are disentangled by the QEC code. (ii) In contrast, for  $\lambda_\alpha > \lambda_\alpha^{\text{critical}}$ , the computer and the environment cannot be disentangled, and decoherence will take place.

This scenario strongly resembles the theory of quantum phase transitions. There are several ways to pursue this analogy; here we present two. First, we just showed that for systems above their critical dimension, the second term of the right-hand side of Eq. (7) is unimportant at asymptotically large scales. In other words, when we calculate the probability  $P(\dots; \alpha, \mathbf{x}_1; \dots)$ , it is a good approximation to replace the operator  $v_\alpha^2(\mathbf{x}_1, \lambda_\alpha^*)$  by its ‘‘mean field’’ (perturbative) value  $\epsilon_\alpha$ . The bare value of  $\lambda_\alpha$  is the only parameter that determines whether QEC can succeed and thus plays a role analogous to temperature. This is precisely what one expects for systems above their upper critical dimension, where the transition has a mean field character [10].

Second, an explicit analogy with a statistical mechanical problem is also possible. Equation (11) can be thought of as the partition function of a gas of fictitious particles, where  $\lambda_\alpha^*$  is the particle fugacity,  $F_\alpha$  creates a particle, and  $F_0$  introduces the vacuum fluctuations. An example is quite illuminating: Consider the simple case of a bosonic bath with  $(\phi, \pi)$  representing the conjugate fields. For a  $D = 1$  computer and a noise model  $V = \lambda \sum_j \cos[\delta\phi(j)] \sigma_j^z$ , Eq. (11) is precisely the partition function of a two-dimensional Coulomb gas. This case, then, has a quantum phase transition in the Kosterlitz-Thouless universality class [16] as a function of  $\delta$ .

On the other hand, going back to Eq. (14), a relevant flow for  $\lambda^*$  indicates that interhypercube correlations produce contributions to the probability function that scale in the same fashion as the intrahypercube part. In this sense, there is no ‘‘de facto’’ separation of scales, or, in other words, the problem is inherently nonperturbative in the coupling with the environment. It is therefore not possible to prove resilience by our arguments.

*Changing  $\dim[F_\alpha]$ .*—We showed previously [12] that small changes in the QEC code can dramatically reduce the effects of correlations between the hypercubes. By simply applying logical NOTs and phase-NOTs in each QEC cycle, the dimension of  $F_\alpha$  becomes larger at the cost of increasing  $\epsilon_\alpha$ . To understand this fact, revisit Eq. (3); for illustration, consider  $\alpha = z$  and apply a logical NOT halfway through the cycle:

$$\hat{v}_z(\mathbf{x}_1, \lambda) \approx i\lambda_z \left[ \int_{\Delta/2}^{\Delta} dt - \int_0^{\Delta/2} dt \right] f_z(\mathbf{x}_1, t).$$

Following the same steps as before, we obtain that  $F_z(\mathbf{x}_1, 0) \propto |\partial_t f_z(\mathbf{x}_1, 0)|^2$ . If  $n$  logical NOTs and phase-

NOTs are performed, we find that  $\dim[F_\alpha] = 2(\delta_\alpha + nz)$ . Therefore, for a given noise model, one can always engineer an irrelevant flow.

*Connection to the AKP results.*—In Ref. [8], a family of long-ranged noise models with interactions between qubits was studied. AKP considered a power law interaction between any two qubits at positions  $\mathbf{x}_1$  and  $\mathbf{x}_2$  of the computer with strength  $\Gamma|\mathbf{x}_1 - \mathbf{x}_2|^{-2\delta}$ . Clearly, one could start from their noise model and use a Hubbard-Stratonovich transformation to arrive at ours. The reverse is also true: Starting from our model, one could integrate out the environment field and arrive at the effective interaction between qubits that AKP considered.

Nevertheless, the two Letters deal with opposite limits of this same model. The crucial difference is the wave velocity of the environment  $v$ . AKP considered the limit of instantaneous interactions  $v \rightarrow \infty$ . This implies that *all* of the qubits of the computer are contained in the volume  $(v\Delta)^{D/z}$ , in contrast to our assumption that *one* qubit is in that volume. Interestingly, the AKP limit leads to an important simplification when combined with QEC: Since errors are detected by QEC, there are no memory effects due to correlations between qubits at different QEC steps. Hence, their error model corresponds in our analysis to a dynamical exponent  $z = 0$ . With this, our criterion for the possibility of resiliency is exactly the same as AKP's, even though the problem and methods used are inherently different. We speculate, therefore, that there is a more general scheme that encompasses both Letters.

*Remarks and conclusions.*—In hindsight, the results that we obtain are dramatically clear and simple. QEC is a perturbative method. Therefore, the threshold condition should be a statement about when a perturbative analysis is valid. In field theories, this is a relatively straightforward question. However, in a quantum computer this is not so obvious due to the presence of the qubits. The remarkable result of QEC is that some of the quantities relevant for the threshold condition depend exclusively on the environment variables. It is, then, possible to derive criteria based only on the field theory that describes the environment.

Our discussion brings to light an interesting parallel with the theory of quantum phase transitions. In fact, we derived a condition that strongly resembles the definition of the upper critical dimension of a quantum problem. When the system is above this dimension, the usual mean field derivation of the threshold theorem is applicable. Hence, as a function of the bare coupling with the environment, there is a transition from the states of the qubits and the environment being disentangled to being entangled.

An important question that remains open is if there is also a lower critical dimension, namely, a criterion for the

impossibility of proving resilience. If the lower and upper critical dimensions are not the same, the intermediate cases would require a proof of resilience substantially different from the *mean field* approach used here. An example of such an approach is the use of the norm of the interaction Hamiltonian by AKP. However, the fact that they found the same critical dimension as we have found suggests that the upper and lower critical dimensions may coincide.

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